Additive set functions and the integral

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Let X be a finite dimensional Euclidean space, $P(X)$ - the power set of X, H - a subsystem of $P(X)$, $f : H \to R$ is a R-valued function defined on H.

A function f defined on X is called an upper (lower) semiadditive provided for all non overlapping sets E_1 and E_2 (i.e. sets E_1 and E_2 have disjoint interiors), with $E_1, E_2 \in H$, $E_1 \cup E_2 \in H$, the folowing inequalities hold:

$$f(E_1 \cup E_2) \le f(E_1) + f(E_2),$$
$$\left(f(E_1 \cup E_2) \ge f(E_1) + f(E_2)\right).$$

A function f defined on X is called additive provided it is both upper and lower semiadditive, i.e. for all non overlapping sets E_1 and E_2, with $E_1, E_2 \in H$, $E_1 \cup E_2 \in H$, the folowing inequalities hold:

$$f(E_1 \cup E_2) = f(E_1) + f(E_2).$$

Let furthermore E_0 be a homogeneous subset of a finite dimensional Euclidean space X, i.e. the set E_0 is Jordan measurable and for any point $P \in E_0$ and its neighbourhood $V(P)$ the set $V(P) \cap E_0$ has a positive Jordan measure.

We will consider the following questions:

1. Construction of an additive function on sets using the given semiadditive functions.
2. Lower and upper integrals as examples of additive functions on sets.
3. Derivative of an additive function on sets.
4. Theorem about the uniqueness of an additive function on sets with a given derivative.
5. Theorem about the integrability of functions which are continuous on a closed homogeneous set.
6. Reconstruction of an additive function on sets from its derivative.

The considered topics are a part of the course “The scientific fundamentals of beginnings of mathematical analysis” for the Master degree program in mathematics. These topics can be used in specific problems of physics and mechanics (for example, a problem about the mass of a body, a problem about an electric charge, a problem about the presssure).