Some Remarks about the Adjoint System

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Summary. Zero properties of solutions of a three-dimensional linear system are discussed. Adjoint system is considered. Solutions of a three-dimensional linear system which satisfy certain boundary conditions and the symmetrical solutions of adjoint system are considered.

1 Introduction

The oscillatory properties of the third-order linear differential equation were extensively investigated in the classical papers by many authors. The linear theory plays an important role in the nonlinear theory. There are given estimates from below of the number of solutions to third-order nonlinear boundary value problems in [3]. The number of solutions depends on the oscillatory properties of a corresponding linear equation.

It is known [2] that if equation $x''' + p(t)x'' + q(t)x' + r(t)x = 0$ with $p \in C^2$, $q \in C^1$, $r \in C$ has a nontrivial solution satisfying the conditions $x(a) = x(b) = x'(b) = 0$, then its adjoint equation has a nontrivial solution which satisfies the conditions $x(a) = x'(a) = x(b) = 0$.

The main purpose of this paper is to generalize this result for solutions of the three-dimensional linear system of the form

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' = a_{21}x_1 + a_{22}x_2 + a_{32}x_3, \\ x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{cases} \tag{1}$$

where the coefficients $a_{ij}(t)$ are continuous functions defined on an interval I unless explicitly stated otherwise.
Setting

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we can write (1) in the form

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{2}$$

Now we shall give some basic concepts and results concerning adjoint system for the reader's convenience. These results can be found in [1].

Consider the system (2). If A^T is the transpose of A, the system

$$\mathbf{y}' = -A^T(t)\mathbf{y}. \tag{3}$$

is called the system adjoint to (2).

Lemma 1.1 [1] *A nonsingular matrix $X(t)$ is a fundamental matrix for* (2) *if and only if $(X^{-1}(t))^T$ is a fundamental matrix for* (3).

2 Preliminaries

In this section we prove some propositions which are used in the main result.
Suppose, that

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & x_{13}(t) \\ x_{21}(t) & x_{22}(t) & x_{23}(t) \\ x_{31}(t) & x_{32}(t) & x_{33}(t) \end{pmatrix} \tag{4}$$

is the fundamental matrix of the system (2).

Lemma 2.1 *If* $\mathbf{x_1}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{31}(t) \end{pmatrix}$ *and* $\mathbf{x_2}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ x_{32}(t) \end{pmatrix}$ *are two linearly independent solutions of* (2) *such that $x_{i1}(a) = x_{i2}(a) = 0$ ($i \in \{1,2,3\}$ is fixed) then any other solution* $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ *for which $x_i(a) = 0$ can be written in the form*

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t),$$

where c_1, c_2 denote arbitrary constants.

Proof. The general solution of (2) is

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + c_3\mathbf{x_3}(t),$$

where c_1, c_2, c_3 denote arbitrary constants.
Since $x_i(a) = 0$ then

$$c_1x_{i1}(a) + c_2x_{i2}(a) + c_3x_{i3}(a) = 0.$$

Since $x_{i1}(a) = x_{i2}(a) = 0$ we get that $c_3 = 0$.
Hence $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$. □

Lemma 2.2 *Suppose that $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are two linearly independent solutions of* (2) *such that $x_{i1}(a) = x_{i2}(a) = 0$ (where $i, j \in \{1,2,3\}$, $i \neq j$ are fixed), and $\mathbf{x}(t)$ is a solution of* (2) *for which $x_i(a) = 0$. If $x_i(\tau) = x_j(\tau) = 0$ ($a \neq \tau$), then the determinant* $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix}$ *is equal to zero.*

Proof. Since $x_i(a) = 0$ then

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t).$$

If $x_i(\tau) = x_j(\tau) = 0$ then

$$\begin{cases} c_1x_{i1}(\tau) + c_2x_{i2}(\tau) = 0, \\ c_1x_{j1}(\tau) + c_2x_{j2}(\tau) = 0. \end{cases}$$

Since $c_1^2 + c_2^2 > 0$ then $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix} = 0.$ □

Lemma 2.3 *Suppose that $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are two linearly independent solutions of* (2) *such that $x_{i1}(a) = x_{i2}(a) = 0$ (where $i, j \in \{1, 2, 3\},\ i \neq j$ are fixed). If the determinant $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix}$ is equal to zero, then there exists a nontrivial solution $\mathbf{x}(t)$ of* (2) *for which $x_i(a) = 0$ and $x_i(\tau) = x_j(\tau) = 0$ $(a \neq \tau)$.*

Proof. If $x_i(a) = 0$ then

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t).$$

Since $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix} = 0$ then there exist such constants α_1, α_2, which are not all zeros $(\alpha_1^2 + \alpha_2^2 > 0)$ that

$$\alpha_1 \begin{pmatrix} x_{i1}(\tau) \\ x_{j1}(\tau) \end{pmatrix} + \alpha_2 \begin{pmatrix} x_{i2}(\tau) \\ x_{j2}(\tau) \end{pmatrix} = 0.$$

Hence there exists a nontrivial solution $\mathbf{x}(t)$ of (2) for which $x_i(a) = 0$ and $x_i(\tau) = x_j(\tau) = 0.$ □

3 Main result

Theorem 3.1 *If $\mathbf{x}(t)$ is a nontrivial solution of the system* (2) *for which $x_i(a) = 0$, $x_i(\tau) = x_j(\tau) = 0$, and $\mathbf{y}(t)$ is a solution of the system* (3) *for which $y_i(a) = y_j(a) = 0$ (where $i, j \in \{1, 2, 3\},\ i \neq j$ are fixed), then $y_i(\tau) = 0$.*

Proof. Since $x_i(a) = 0$ then by the lemma 2.1

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t),$$

where $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are two linearly independent solutions of (2) for which $x_{i1}(a) = x_{i2}(a) = 0$ and c_1, c_2 denote arbitrary constants.

Since $x_i(\tau) = x_j(\tau) = 0$ then (by the lemma 2.2) $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix} = 0.$

By the lemma 1.1

$$\mathbf{y}(t) = \frac{1}{\det X(t)} \begin{pmatrix} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \\ -\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \\ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

is the solution of the system (3).
More over $y_i(a) = y_j(a) = 0$.
Since $\begin{vmatrix} x_{i1}(\tau) & x_{i2}(\tau) \\ x_{j1}(\tau) & x_{j2}(\tau) \end{vmatrix} = 0$ then $y_i(\tau) = 0$. $\square$

Corollary 3.1 *If system* (2) *has a nontrivial solution which satisfies the conditions* $x_1(a) = 0,\ x_1(\tau) = x_2(\tau) = 0$, *then the adjoint system* (3) *has a nontrivial solution which satisfies the conditions* $y_1(a) = y_2(a) = 0,\ y_1(\tau) = 0$.

Corollary 3.2 *If system* (2) *has a nontrivial solution which satisfies the conditions* $x_1(a) = 0,\ x_1(\tau) = x_3(\tau) = 0$, *then the adjoint system* (3) *has a nontrivial solution which satisfies the conditions* $y_1(a) = y_3(a) = 0,\ y_1(\tau) = 0$.

Remark 3.1. 4 analogous statements can be formulated.
Remark 3.2. Theorem 3.1 has certain geometrical interpretation. Suppose that the solution of the system (2) is a point $\big(x_1(t), x_2(t), x_3(t)\big)$ in the three-dimensional space at every moment of time t. Suppose that the nontrivial solution (point) of the system (2) at the time moment $t = a$ is located on some coordinate plane α and at the time moment $t = \tau$ is located on some coordinate axis l which belongs to coordinate plane α. Then the adjoint system (3) has the nontrivial solution (point) which at the time moment $t = a$ is located on the coordinate axis l and at the time moment $t = \tau$ is located on the coordinate plane α.

4 Example

Consider the system

$$\begin{cases} x_1' = x_2, \\ x_2' = -x_1, \\ x_3' = (\cos t + 3t^2 \sin t)x_1 + (3t^2 \cos t - \sin t)x_2. \end{cases} \tag{5}$$

System's (5) the fundamental matrix is

$$\begin{pmatrix} \sin t & \cos t & -\cos t \\ \cos t & -\sin t & \sin t \\ t^3 & t & 1 - t \end{pmatrix}. \tag{6}$$

The system

$$\begin{cases} y_1' = y_2 - (\cos t + 3t^2 \sin t)y_3, \\ y_2' = -y_1 + (\sin t - 3t^2 \cos t)y_3, \\ y_3' = 0 \end{cases} \tag{7}$$

is adjoint to (5).

System's (7) the fundamental matrix is

$$\begin{pmatrix} \sin t & (1-t)\cos t - t^3 \sin t & -t\cos t - t^3 \sin t \\ \cos t & (t-1)\sin t - t^3 \cos t & t \sin t - t^3 \cos t \\ 0 & 1 & 1 \end{pmatrix}. \tag{8}$$

Consider the solution of the system (5)

$$\mathbf{x}(t) = \begin{pmatrix} \sin t \\ \cos t \\ t^3 \end{pmatrix}$$

and the solution of the system (7)

$$\mathbf{y}(t) = \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}.$$

Since

$$x_1(\pi) = 0, \quad x_1(0) = 0, \quad x_3(0) = 0,$$

then (by the theorem 3.1)

$$y_1(\pi) = 0, \quad y_3(\pi) = 0, \quad y_1(0) = 0.$$

As we can see, the equalities are valid.

References

[1] P. Hartman, Ordinary Differential Equations, John Wiley and Sons, New York-London-Sydney, 1964.

[2] N.V. Azbelev and Z.B. Caljuk, On the question of distribution of zeros of solutions of linear differential equations of third order, Mat. Sb. 51 (93) (1960), 475-486; English transl., Amer. Math. Soc. Transl. (2) 42 (1964), 233-245.

[3] F. Sadyrbaev, Multiplicity results for third order two-point boundary value problems, In the paper collection: Proceedings of the University of Latvia. Mathematics. Differential equations, vol. 616, LU, 1999, 5 - 16.

С. Смирнов. Замечания о сопряжённой системе.

Аннотация. Обсуждаются свойства нулей решений линейной трёхмерной системы. Рассматривается сопряжённая система. Рассматриваются решения линейной трёхмерной системы удовлетворяющие определённым краевым условиям и симметричные им решения сопряжённой системы.

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S. Smirnovs. Piezīmes par saistīto sistēmu.

Anotācija. Tiek apspriestas trīs-dimensiju sistēmas atrisinājumu nulļu īpašības. Tiek apskatīta saistīta sistēma. Tiek apskatīti trīs-dimensiju sistēmas atrisinājumi kuri apmierina noteiktus robežnosācījumus un tiem simetriskie saistītas sistēmas atrisinājumi.

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