On three-point boundary value problems

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Summary. We provide the conditions for solvability of the boundary value problem $x'' = g(t,x,x') + h(t,x,x'),\ p\,x(0) + x'(0) = 0,\ Hx = 0.$

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1 Introduction

In the work [1] the existence of a solution to the boundary value problem

$$\begin{aligned} &x'' = g(t,x,x') + h(t,x,x'), \quad t \in I := [0,1], \\ &p\,x(0) + x'(0) = 0, \quad x(1) = \alpha x(\eta), \quad \alpha \le 0,\ 0 < \eta < 1 \end{aligned}$$

was proved under the assumptions that

$$\begin{aligned} &x' g(t,x,x') \le 0, \\ &|h(t,x,x')| \le a(t)|x| + b(t)|x'| + u(t)|x|^r + v(t)|x'|^k + e(t), \quad 0 \le r,k < 1, \\ &(|p| + a_1)e^{b_1} < 1, \quad a_1 = \|a(t)\|_1, \quad b_1 = \|b(t)\|_1. \end{aligned}$$

Moreover, the existence of a positive solution to the boundary value problem

$$x'' + g(t)f(x,x') = 0, \quad x'(0) = 0,\ x(1) = \alpha x(\eta), \quad \alpha, \eta \in (0,1),$$

where $g \in C(I,[0,+\infty)),\ f \in C([0,+\infty) \times (-\infty,0],[0,+\infty))$, was obtained. Three- and multipoint boundary value problems were considered in the works [2] - [10].

Our purpose in this paper is twofold. First, we show that the conditions $(p_+ + a_1)e^{b_1} < 1$, where $p_+ = \max\{0,p\}$, or $(p_+ + a_1)e^{b_1} = 1$ and $a_1 + b_1 > 0$ imply the existence of a solution to the problem (1). It is possible to construct counterexample which shows that for $a_1 = b_1 = 0$ and $p = 1$ the problem has not a solution. Counterexample can be constructed also which shows that the problem has not a solution if the condition $(p_+ + a_1)e^{b_1} > 1$ holds.

Second, we provide conditions for existence of a positive solution. These conditions differs slightly from the conditions of Theorem 3.2 in [1] and cannot be improved. This is confirmed by construction the respective examples.

2 Setting of a problem

Consider the problem

$$x'' = g(t, x, x') + h(t, x, x'), \quad t \in I, \tag{1}$$

$$p(x(0)) + x'(0) = 0, \; Hx = 0, \tag{2}$$

where functions g and h satisfy the Caratheodory conditions, $p \in C(R, R)$ and $H \in C(C^1(I, R), R)$.

We assume that the following conditions hold:

(1) $x' \, g(t, x, x') \le 0, \quad (t, x, x') \in I \times R^2$;

(2) There exist functions $a, b, c, d \in L_1(I, [0, +\infty))$ such that for any $\varepsilon > 0$ there exists $e \in L_1(I, [0, +\infty))$ such that

$$|h(t, x, x')| \le (a(t) + \varepsilon c(t)) \, |x| + (b(t) + \varepsilon d(t))|x'| + e(t),$$

holds for any $(t, x, x') \in I \times R^2$;

(3) There exist $p_+, q \in [0, +\infty)$ such that

$$p\,(N) \ge p_+ N - q, \quad N \le 0, \; p(N) \le p_+ N + q, \; N \ge 0.$$

(4) There exist $N_0 \ge 0$ such that for any $x \in C^1(I, R)$

$$(\forall t \in I)(x(t) > N_0) \Rightarrow Hx \ge 0, \; (\forall t \in I)(x(t) < -N_0) \Rightarrow Hx \le 0.$$

Lemma 2.1 *Let $a, b, e \in L_1(I, [0, +\infty))$, $x_0, x_1 \in R$. Suppose that the estimate*

$$|h(t, x, x')| \le a(t)|x| + b(t)|x'| + e(t), \quad (t, x, x') \in I \times R^2$$

holds. Then a solution x of the Cauchy problem

$$x'' = g(t, x, x') + h(t, x, x'), \quad x(0) = x_0, \; x'(0) = x_1$$

extends to the interval I and satisfies the estimates $-y \le x \le y$, where y is a solution to the Cauchy problem

$$y'' = a(t)y + b(t)y' + e(t), \quad y(0) = |x_0|, \; y'(0) = |x_1|.$$

Proof. Let us show that boundedness of a solution $x(t)$ in I implies boundedness of a derivative $x'(t)$. Suppose the contrary is true. Consider the case

$$x : [0, \tau) \to R, \; \tau \in (0, 1], \; \sup\{|x(t)| : \; t \in [0, \tau)\} < +\infty, \; \lim_{t \to \tau} x'(t) = +\infty.$$

Let $t_1 \in (0, \tau)$ be such that

$$x'(t) > 0, \; t \in (t_1, \tau), \; \int_{t_1}^{\tau} b(t)\, dt < \frac{1}{2}.$$

The inequality

$$\begin{aligned} x''(t) = \; & g(t, x(t), x'(t)) + h(t, x(t), x'(t)) \le h(t, x(t), x'(t)) \\ \le \; & a(t)|x(t)| + b(t)|x'(t)| + e(t), \quad t \in (t_1, \tau) \end{aligned}$$

implies that

$$x'(t_2) - x'(t_1) \le \int_{t_1}^{t_2} (a(t)|x(t)| + e(t))\, dt + \frac{x'(t_2)}{2}$$

for $t_2 \in (t_1, \tau)$ such that $x'(t_2) = \max\{x'(t) : t \in [t_1, t_2]\}$. This inequality is impossible for sufficiently large $x'(t_2)$. The case of $\lim_{t\to\tau} x'(t) = -\infty$ can be considered in a similar way.

Choose $\varepsilon > 0$ and let y_ε be a solution of the Cauchy problem

$$y'' = a(t)y + b(t)y' + e(t), \quad y(0) = |x_0|,\ y'(0) = |x_1| + \varepsilon.$$

We wish to show that the inequalities $-y_\varepsilon \le x \le y_\varepsilon$ hold for any $\varepsilon > 0$. Then the estimates $-y \le x \le y$ are evident. Consider the case of existing $\tau \in (0, 1)$ such that $y_\varepsilon'(\tau) = x'(\tau)$ and

$$-\, y_\varepsilon'(t) < x'(t) < y_\varepsilon'(t), \quad t \in (0, \tau). \tag{3}$$

Choose $t_1 \in (0, \tau)$ such that $x'(t) > 0,\ t \in (t_1, \tau)$. Then

$$\begin{aligned} &x''(t) = g(t, x(t), x'(t)) + h(t, x(t), x'(t)) \le h(t, x(t), x'(t)) \\ &\le a(t)|x(t)| + b(t)x'(t) + e(t) \le a(t)y_\varepsilon(t) + b(t)y_\varepsilon'(t) + e(t) = y_\varepsilon''(t), \quad t \in (t_1, \tau). \end{aligned}$$

Therefore either $x'(\tau) - x'(t_1) \le y_\varepsilon'(\tau) - y_\varepsilon'(t_1)$ or $y_\varepsilon'(t_1) \le x'(t_1)$, which contradicts the inequalities (3).

The case of $-y_\varepsilon'(\tau) = x'(\tau)$ can be considered in a similar way. □

Lemma 2.2 *Let $a, b, e \in L_1(I, [0, +\infty))$, $N > 0$ and the estimate*

$$|h(t, x, x')| \le a(t)|x| + b(t)|x'| + e(t), \quad (t, x, x') \in I \times R^2$$

hold. If $(p_+ + a_1 + \frac{e_1+q}{N})\, e^{b_1} < 1$, where $e_1 = \|e(t)\|_1$, then a solution x of the Cauchy problem

$$x'' = g(t, x, x') + h(t, x, x'), \quad x(0) = -N,\ x'(0) = -p(-N)$$

satisfies the estimate

$$x(t) \le -N + t(Np_+ + Na_1 + e_1 + q)\, e^{b_1}, \quad t \in I. \tag{4}$$

Proof. Consider the case of $x'(t) \ge 0,\ t \in I$. Then

$$\begin{aligned} &x''(t) = g(t, x(t), x'(t)) + h(t, x(t), x'(t)) \le h(t, x(t), x'(t)) \\ &\le -a(t)|x(t)| + b(t)x'(t) + e(t) \le b(t)x'(t) + a(t)N + e(t), \quad t \in I. \end{aligned}$$

A solution $y(t)$ of the Cauchy problem

$$y'(t) = b(t)y(t) + a(t)N + e(t), \quad y(0) = p_+N + q \tag{5}$$

by comparison theorem satisfies the inequality $x'(t) \le y(t),\ t \in I$. The explicit formula for a solution of the problem (5) is

$$y(t) = (p_+N + q) \exp \int_0^t b(s)\, ds + \int_0^t (a(s)N + e(s)) \exp(\int_0^t b(\xi)\, d\xi)\, ds. \tag{6}$$

It follows from (6) that

$$y(t) \le (Np_+ + q + Na_1 + e_1)e^{b_1}, \quad t \in I. \tag{7}$$

The estimate (4) follows.

Consider the case of $x(t) \ge -N,\ t \in I$. Let $T = \{t \in (0,1) :\ x'(t) > 0\}$. It is clear that T is an open set, which is a sum of disjoint open intervals. Let (t_1, t_2) be such an interval. If $t_1 = 0$, then the proof is similar to the previous one. If $t_1 > 0$, the $x'(t_1) = 0$ and the estimate in $[t_1, t_2]$ can be obtained like previously. Hence the estimate (7) follows.

Consider the case of $\min\{x(t) :\ t \in I\} < -N$. Let $x(t_1) = -N_1 < -N$. Then the estimate

$$x(t) \le -N_1 + (t - t_1)(N_1a_1 + e_1)e^{b_1} \le -N + (t - t_1)(N_1a_1 + e_1)e^{b_1}, \quad t \in [t_1, t_2].$$

The estimate (4) follows. □

Lemma 2.3 *Suppose there exist two solutions x_1 and x_2 of equation (1) such that*

$$p(x_1(0)) + x_1'(0) = p(x_2(0)) + x_2'(0) = 0, \quad Hx_1Hx_2 \le 0.$$

Then there exist a solution of the problem (1), (2).

Proof. Let $x_1(0) \le x_2(0)$ and denote by S a set of solutions of equation (1) which satisfy the conditions

$$p(x(0)) + x'(0) = 0, \quad x_1(0) \le x(0) \le x_2(0).$$

Since S is a connected set and $x_1, x_2 \in S$ there exists $x \in S$ such that $Hx = 0$. □

Theorem 2.1 *Suppose that $(p_+ + a_1)e^{b_1} < 1$. Then there exists a solution to the problem (1), (2).*

Proof. Let $\varepsilon > 0$ be such that the inequality

$$(p_+ + a_1 + \varepsilon c_1)e^{b_1 + \varepsilon d_1} < 1$$

holds, where $c_1 = \|c(t)\|_1$ and $d_1 = \|d(t)\|_1$. Let $e(t)$ be a function from the condition 2. Choose $N > 0$ such that the inequalities

$$(p_+ + a_1 + \varepsilon c_1 + \frac{e_1 + q}{N})e^{b_1 + \varepsilon d_1} < 1,$$

$$N - (N(p_+ + q_1 + \varepsilon c_1) + e_1 + q)e^{b_1 + \varepsilon d_1} > N_0$$

hold, where $e_1 = \|e(t)\|_1$. The estimate

$$x_1(t) \le -N + t(N(p_+ + a_1 + \varepsilon c_1) + e_1 + q)e^{b_1 + \varepsilon d_1} < -N_0, \quad t \in I$$

follows from Lemma 2.2 for a solution x_1 of the Cauchy problem

$$x'' = g(t, x, x') + h(t, x, x'), \quad x(0) = -N,\ x'(0) = -p(-N).$$

Similarly the estimate $x_2(t) > N_0,\ t \in I$ can be obtained for a solution x_2 of the Cauchy problem

$$x'' = g(t, x, x') + h(t, x, x'), \quad x(0) = N,\ x'(0) = -p(N).$$

Then solvability of the problem (1), (2) follows from Lemma 2.3. □

Theorem 2.2 *Suppose that* $(p_+ + a_1)e^{b_1} = 1$ *and* $a_1 + b_1 > 0$. *Then there exists a solution to the problem (1), (2).*

Proof. Consider the case of $a_1 > 0$. Pick $\tau \in (0,1)$ such that $\int_0^\tau a(t)\,dt = \frac{a_1}{2}$. It follows from the condition 2 and Lemma 2.2 that

$$x(t) \le -N + t(N(p_+ + \frac{a_1}{2} + \varepsilon c_1) + e_1 + q)e^{b_1}, \quad t \in [0,\tau],$$

$$\begin{aligned} x(t) &\le -N + \tau(N(p_+ + \tfrac{a_1}{2} + \varepsilon c_1) + e_1 + q)e^{b_1} \\ &+(t-\tau)(N(p_+ + a_1 + \varepsilon c_1) + e_1 + q)e^{b_1} \\ &= -N + t(N(p_+ + \tfrac{a_1}{2} + \varepsilon c_1) + e_1 + q)e^{b_1} - \tau N a_1 \tfrac{e^{b_1}}{2}, \quad t \in [\tau, 1]. \end{aligned}$$

Choose $\varepsilon > 0$ and $N > 0$ such that the inequalities

$$\varepsilon c_1 - \tau\frac{a_1}{2} < 0, \quad (N(\varepsilon c_1 - \tau\frac{a_1}{2}) + e_1 + q)e^{b_1} < -N_0.$$

Solutions x_1 and x_2 can be constructed as in the proof of Theorem 2.1. Solvability of the boundary value problem (1), (2) follows from Lemma 2.3. The cases of $a_1 = 0$ and $b_1 > 0$ can be considered as above. □

Examples. The examples below show that the estimates of theorems 2.1 and 2.2 are best possible.

Let $a_1 = b_1 = 0$ and $p(N) = N$. Then a solution of the Cauchy problem

$$x'' = 2, \quad x(0) = c, \ x'(0) = -c$$

has the form $x(t) = t^2 - ct + c$ and $x(1) = 1$. Therefore the BVP

$$x'' = 2, \quad x(0) + x'(0) = 0, \ x(1) = 0$$

has not solutions.

Let $(p_+ + a_1)e^{b_1} > 1$, $\delta \in (0, \frac{1}{4})$ and $g(t,x') = \max\{0, -2\delta^{-1}x'^3\}$, $t \in [0,\delta)$, $g(t,x') = 0$, $t \in [\delta, 1]$, $h(t,x,x') = 0$, $t \in [0,\delta) \cup [4\delta, 1]$,
$h(t,x,x') = \delta^{-1}$, $t \in [\delta, 2\delta)$, $h(t,x,x') = \max\{0, -a_1\delta^{-1}x\}$, $t \in [2\delta, 3\delta)$, $h(t,x,x') = b_1\delta^{-1}x'$, $t \in [3\delta, 4\delta)$.

Then a solution of the Cauchy problem

$$x'' = g(t,x') + h(t,x,x'), \quad x(0) = c, \ x'(0) = -p_+c$$

satisfies the condition $x(1) > 0$. Therefore the BVP

$$x'' = g(t,x') + h(t,x,x'), \quad p_+x(0) + x'(0) = 0, \ x(1) = 0$$

has not a solution.

Theorem 2.3 *Let* y *be a solution of the Cauchy problem*

$$y'' = -a(t)y + b(t)y', \quad y(0) = 1, \ y'(0) = -p_+.$$

If $y(t) > 0$ *for* $t \in I$ *then the BVP (1), (2) has a solution.*

Proof. Choose $\varepsilon > 0$ such that for $e_* \in L(I,[0,+\infty))$, $\|e_*\|_1 < \varepsilon$ a solution of the Cauchy problem

$$y_\varepsilon'' = -(a(t)+\varepsilon c(t))y_\varepsilon + (b(t)+\varepsilon d(t))y_\varepsilon' - e_*(t), \quad y_\varepsilon(0) = 1,\ y_\varepsilon'(0) = -p_+ - \varepsilon$$

satisfies the inequality $y_\varepsilon > \frac{y(1)}{2}$. The constants c and d are the same as appaer il the condition 2. For ε given we find the appropriate $e(t)$ using the condition 2. Consider solutions of the following problems

$$x_1'' = g(t,x_1,x_1') + h(t,x_1,x_1'), \quad x_1(0) = N,\ x_1'(0) = -p(N),\ N \in (\frac{q}{\varepsilon}, +\infty),$$

$$y_*'' = -(a(t)+\varepsilon c(t))y_* + (b(t)+\varepsilon d(t))y_*' - \frac{e(t)}{x_1(t)}, \quad y_*(0) = 1,\ y_*'(0) = -p_+ - \varepsilon.$$

Let us show that $z = x_1'y_* - y_*'x_1 \geq 0$. Indeed, $z(0) = x_1'(0)y_*(0) - y_*'(0)x_1(0) \geq -p_+N - q + (p_+ + \varepsilon)N > 0$. Let $t_1 = \sup\{t \in I:\ (\forall \tau \in [0,t])(z(\tau) \geq 0)\}$. Notice that if $z \geq 0$ in the interval $[0,t_1]$ then $\frac{x_1'}{x_1} \geq \frac{y_*'}{y_*}$ and $(\ln x_1)' \geq (\ln y_*)'$. Integration from 0 to t yields $\ln x_1(t) - \ln x_1(0) \geq \ln y_*(t) - \ln y_*(0)$. Therefore $\frac{x_1(t)}{x_1(0)} \geq \frac{y_*(t)}{y_*(0)}$ or $x_1(t) \geq Ny_*(t)$. If $t_1 = 1$, then $z \geq 0$. Let $t_1 \in (0,1)$. If $x_1(t_1) \geq 0$, then $z(t_1) = x_1'(t_1)y_*(t_1) - y_*'(t_1)x_1(t_1) \geq x_1'(t_1)y_*(t_1) + (p_+ + \varepsilon)x(t_1) > 0$, and this contradicts the definition of t_1. In case of $x_1'(t_1) < 0$ choose $t_2 \in (t_1,1)$ such that $x_1(t) \leq 0,\ t \in (t_1,t_2)$. Then the relations

$$\begin{aligned} x_1'' &= g(t,x_1,x_1') + h(t,x_1,x_1') \geq h(t,x_1,x_1') \\ &\geq -(a(t)+\varepsilon c(t))x_1 + (b(t)+\varepsilon d(t))x_1' - e(t), \end{aligned}$$

$$y_*'' = -(a(t)+\varepsilon c(t))y_* + (b(t)+\varepsilon d(t))y_*' - \frac{e(t)}{x_1(t)}$$

hold in the interval $[t_1,t_2]$. A constant N here must be such that $N > \|2\frac{e(t)}{y(1)}\|_1\varepsilon^{-1}$. Multiplying the inequality above by y_*, the equality above by x_1 and subtracting the latter from the first one obtains that

$$\begin{aligned} &x_1''y_* - y_*''x_1 \geq (b(t)+\varepsilon d(t))(x_1'y_* - y_*'x_1) - e(t)y_* + e(t) \\ &\geq (b(t)+\varepsilon d(t))(x_1'y_* - y_*'x_1). \end{aligned}$$

Hence $x_1''y_* - y_*''x_1 = (x_1'y_* - y_*'x_1)' = z' \geq (b(t)+\varepsilon d(t))z$ and $z(t_1) \geq 0$. By the comparison theorem $z(t) \geq 0,\ t \in [t_1,t_2]$, which contradicts the definition of t_1. Using the same type arguments one yields from $z \geq 0$ that $x_1 \geq Ny_*$. If $Ny_*(1) > N_0$, then $x_1 > N_0$. Similarly x_2 can be found as a solution of the BVP

$$x_2'' = g(t,x_2,x_2') + h(t,x_2,x_2'), \quad x_2(0) = -N,\ x_2'(0) = -p(-N).$$

It is clear that $x_2 < -N_0$. Solvability of the BVP (1), (2) follows from Lemma 2.3. □

Theorem 2.6 of the work [1] contains the following conditions for solvability of the BVP

$$\begin{gathered} x'' = f(t,x,x') + e(t), \quad f = g + h, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \quad \alpha \in R \setminus \{\eta^{-1}\}, \quad \eta \in (0,1): \end{gathered} \tag{8}$$

1. $(\exists M_1 > 0)(|p| > M_1 \Rightarrow f(t,x,p) + e(t) \neq 0)$,
2. $(\exists M_2 > 0)(|p| > M_2 \Rightarrow p f(0,0,p) \geq 0)$,
3. $p\, g(t,x,p) \leq 0$,
4. $|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t), \quad a,b,u,v,c \in L_1,\ 0 \leq r,k \leq 1$,
5. $(C_0 + a_1)e^{b_1} < 1, \quad a_1 = \|a(t)\|_1,\ b_1 = \|b(t)\|_1$,

where

$$C_0 = \begin{cases} 0, & \alpha \leq 1, \\ \dfrac{\alpha - 1}{\alpha(1-\eta)}, & 1 < \alpha < \eta^{-1}, \\ \dfrac{1}{\alpha\eta}, & \alpha > \eta^{-1}. \end{cases}$$

The following example shows that formulation of theorem 2.6 in [1] needs to be made more precise.

Consider

$$x'' = \min\{0, -6l^3 x'^3 \max\{0, x\}\} - 1, \quad l > 0,$$
$$x(0) = 0, \quad x(1) = \alpha x(\eta).$$

Suppose that η is fixed. Then for any $\varepsilon > 0$ there exists l such that the problem (8) has a solution only for $\alpha \in (\eta^{-1}, \eta^{-2} + \varepsilon)$.

3 Existence of a positive solution

Consider the boundary value problem

$$x'' + g(t)f(x,x') = 0, \quad x'(0) = 0,\ x(1) = \alpha x(\eta), \tag{9}$$

where $g \in L_1(I, [0,+\infty)),\ f \in C([0,+\infty) \times (-\infty, 0], [0,+\infty))$ and $\alpha, \eta \in (0,1)$.

Theorem 3.1 *Suppose the conditions*
1. $\int_0^1 g(t)\,dt = 1$,
2. $(\forall x_1, x_2 \in [0,+\infty))(\forall x_1', x_2' \in (-\infty, 0])(x_1 \leq x_2 \wedge x_1' \leq x_2' \Rightarrow f(x_1,x_1') \leq f(x_2,x_2'))$,
3. $f(0,0) > 0$,
4. $(\exists H > 0)(\forall x \geq H)(f(x,0) \leq Dx),\ D \in (0, \frac{1-\alpha}{1-\alpha\eta}]$
are fulfilled.
Then the BVP (9) has a positive solution.

Proof. Define $f(x,x')$ for $x < 0$ by the formula $f(x,x') = f(0,x')$. Consider solutions x_N of the Cauchy problems

$$x'' + g(t)f(x,x') = 0, \quad x(0) = N,\ x'(0) = 0$$

for $N > 0$. If N is sufficiently small then the graph of a solution x_N crosses the t-axis. Consider behavior of x_N for N great. The estimates $x_N' \geq -ND$ and $x_N(t) \geq N - NDt$, $t \in I$ can be obtained repeating the arguments of the proof of Lemma 2.2. Therefore $x_N(t) > 0$ for N sufficiently large. Then a solution x_N exists such that $x_N(1) = 0$. If uniqueness of solutions of initial value problems is presupposed then a solution of the BVP (9) exists for $\alpha \in (0, \frac{1-D}{1-D\eta}]$. Indeed, the ratio $\frac{x_N(1)}{x_N(\eta)}$ satisfies the inequality

$$\begin{aligned}\frac{x_N(1)}{x_N(\eta)} &\geq x_N(\eta) - ND\tfrac{1-\eta}{x_N(\eta)} = 1 - ND\tfrac{1-\eta}{x_N(\eta)} \\ &\geq 1 - ND\frac{1-\eta}{N(1-D\eta)} = \frac{1-D}{1-D\eta}\end{aligned}$$

under the condition that $x'_N \geq -ND$. By using the approximation procedure of $f(x, x')$ by the functions $f_n(x, x')$, which satisfy the Lipschitz condition and the conditions 2 to 4 of the theorem, one can obtain a sequence of positive solutions x_n of boundary value problems

$$x'' + g(t)f_n(x, x') = 0, \quad x'(0) = 0,\ x(1) = \alpha x(\eta),$$

which converges to a positive solution of the BVP (9).

Remark 1. The condition $f(0, 0) > 0$ is equivalent to the condition

$$(\exists H_1 > 0)(|x| + |x'| \leq H_1 \Rightarrow |f(x, x')| \geq C(|x| + |x'|), \quad C > 0$$

of the theorem 3.2 of the work [1]. Indeed,

$$0 < CH_1 \leq |f(0, -H_1)| = f(0, -H_1) \leq f(0, 0).$$

The opposite implication follows from the continuity of f.

The condition $(\exists H > 0)(\forall x \geq H)(f(x, 0) \leq Dx)$ is equivalent to the condition

$$(\exists H_2 > 0)(|x| + |x'| \geq H_2 \Rightarrow |f(x, x')| \leq D(|x| + |x'|), \quad b > 0$$

of the theorem 3.2 of the work [1]. Indeed,

$$|f(x, x')| = f(x, x') \leq f(x, 0) \leq bx \leq D(|x| + |x'|).$$

The opposite implication is evident.

Example. Consider

$$x'' = g_\delta(t) \max\{1 - D, Dx\}, \quad x(0) = 0,\ x(1) = \alpha x(\eta), \tag{10}$$

where $\delta \in (0, 1)$, $g_\delta(t) = \delta^{-1}$ for $t \in [0, \delta)$, $g(t) = 0$ for $t \in [\delta, 1]$, $D, \eta \in (0, 1)$. Let us show that for any $\varepsilon > 0$ there exists δ such that the problem (10) has not a solution for $\alpha = \frac{1-D}{1-D\eta} + \varepsilon$. Indeed, for sufficiently small δ a solution of the Cauchy problem

$$x'' = g_\delta(t) \max\{1 - D, Dx\}, \quad x(0) = 1,\ x'(0) = 0$$

is close to $1 - Dt$.

Remark 2. If g is fixed then the estimates for D and α can be improved. Let λ_1 be the first eigenvalue of the BVP

$$x'' + \lambda g(t)x = 0, \quad x'(0) = 0,\ x(1) = 0,$$

$D \in (0, \lambda_1)$ and y be a solution of the Cauchy problem

$$y'' + g(t)Dy = 0, \quad y(0) = 1,\ y'(0) = 0.$$

Then there exists a positive solution of the BVP (9) for $\alpha \in (0, \frac{y(1)}{y(\eta)})$. Indeed, let x stand for a solution of the BVP

$$x'' + g(t)f(x, x') = 0, \quad x(1) \geq H,\ x'(0) = 0.$$

Then

$$x'' + g(t)Dx \geq 0, \quad y'' + g(t)Dy = 0.$$

One gets multiplying the first inequality by y, ten the second equality by x and subtracting the second from the first that $x''y - y''x = (x'y - y'x)' \geq 0$. Integration from 0 to t yields $x'(t)y(t) - y'(t)x(t) \geq 0$ or $\frac{x'}{x} \geq \frac{y'}{y}$. Hence $(\ln x)' \geq (\ln y)'$. Integrating this inequality from η to 1 one obtains $\ln x(1) - \ln x(\eta) \geq \ln y(1) - \ln y(\eta)$. Therefore $\frac{x(1)}{x(\eta)} \geq \frac{y(1)}{y(\eta)}$, which proves the assertion.

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А. Лепин. О трехточечной краевой задаче.

Аннотация. Указаны условия существования решения краевой задачи $x'' = g(t, x, x') + h(t, x, x'),\ \ p\,x(0) + x'(0) = 0,\ \ Hx = 0.$

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A. Lepins. Par vienu trispunktu robežproblēmu.

Anotācija. Tiek doti robežproblēmas $x'' = g(t, x, x') + h(t, x, x'),\ \ p\,x(0) + x'(0) = 0,$ $Hx = 0$ atrisinājuma eksistences nosacījumi.

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