Time map formulae and their applications

A. Gritsans, F. Sadyrbaev

Summary. A set of time map $U(\alpha, \lambda)$ formulae for the problem $x'' + \lambda f(x) = 0$ (i), $x(a) = 0,\ x'(a) = \alpha > 0$ (ii) is provided. We derive conditions for monotonicity and convexity of U in both arguments and apply them a) to obtain some multiplicity results for the Dirichlet problem (i), $x(a) = x(b) = 0$ (iii) and b) to get assertions about similarity of the Dirichlet (Neumann) nonlinear Fučík type spectra to the classical Fučík spectrum.

MSC: 34B15
Keywords: nonlinear spectra, time map, Fučík spectrum.

1 Introduction

Consider equation

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{1}$$

or, what is the same,

$$x'' = \begin{cases} -\lambda f(x) & \text{if} \quad x \geq 0, \\ \mu g(-x) & \text{if} \quad x < 0, \end{cases} \tag{2}$$

together with the Dirichlet boundary conditions

$$x(a) = 0, \quad x(b) = 0 \tag{3}$$

and the normalization condition

$$x'(a) = \alpha > 0, \tag{4}$$

under the assumptions that f un g are continuous positive valued functions such that $f(0) = g(0) = 0$.

To formulate next two theorems we assume that f satisfies the following condition:
(A1) A first zero $t_1(\gamma)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(a) = 0,\ u'(a) = \gamma$$

is finite for any $\gamma > 0$.

A similar property can be assigned to g.

(A2) A first zero $\tau_1(\delta)$ of a solution to the Cauchy problem

$$v'' = -g(v), \quad v(a) = 0,\ v'(a) = \delta$$

is finite for any $\delta > 0$.

Let us recall (slighty modificated) the main result in [3] for the case of the Dirichlet problem.

Theorem 1.1 *Let the conditions **(A1)** and **(A2)** hold with respect to the functions f and g. The Fučík type spectrum for the Dirichlet problem* (1), (3) *with the normalization condition* (4) *is given by the relations* ($i = 1, 2, \ldots$)*:*

$$F_0^+ = \left\{(\lambda,\mu):\ \lambda \text{ is a solution of } \frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) = b-a, \quad \mu \geq 0\right\},$$

$$F_0^- = \left\{(\lambda,\mu):\ \lambda \geq 0,\ \ \mu \text{ is a solution of } \frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) = b-a\right\},$$

$$F_{2i-1}^+ = \left\{(\lambda,\mu):\ i\,\frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) + i\,\frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) = b-a\right\},$$

$$F_{2i-1}^- = \left\{(\lambda,\mu):\ i\,\frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) + i\,\frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) = b-a\right\},$$

$$F_{2i}^+ = \left\{(\lambda,\mu):\ (i+1)\,\frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) + i\,\frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) = b-a\right\},$$

$$F_{2i}^- = \left\{(\lambda,\mu):\ (i+1)\,\frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) + i\,\frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) = b-a\right\}.$$

Similar result is valid ([2]) for the Neumann boundary conditions

$$x'(a) = 0, \quad x'(b) = 0, \tag{5}$$

where the normalization condition is

$$\left|x'(z)\right| = \alpha > 0,\ \ \{z \in (a,b):\ x(z) = 0\}. \tag{6}$$

Let us recall the above mentioned result in a slighty modified form.

Theorem 1.2 *Let the conditions **(A1)** and **(A2)** hold with respect to the functions f and g. The Fučík type spectrum for the Neumann problem* (1), (5) *with the normalization condition* (6) *is given by the relations* ($i = 1, 2, \ldots$)*:*

$$F_i^{\pm} = \left\{(\lambda,\mu):\ i\,\frac{1}{2}\,\frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) + i\,\frac{1}{2}\,\frac{1}{\sqrt{\mu}}\, \tau_1\left(\frac{\alpha}{\sqrt{\mu}}\right) = b-a\right\}.$$

Both Theorem 1.1 and 1.2 involve the function

$$U(\alpha,\lambda) = \frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right),$$

which has the following meaning: for given $\alpha > 0$ and $\lambda > 0$ $U(\alpha,\lambda)$ is the distance from $t = a$ to the first (after a) zero of a solution to the initial value problem

$$x'' + \lambda f(x) = 0 \tag{7}$$

$$x(a) = 0, \quad x'(a) = \alpha > 0, \tag{8}$$

in other words, the function $U(\alpha, \lambda)$ is the first zero function for a solution of the problem (7), (8). It is possible to study the function t_1 and then to make consequences about properties of the function U, but we prefer to study the function U directly as a function of two arguments, looking for expressions of the first and second order partial derivatives.

If f and g are linear then (1) becomes the classical Fučík equation

$$x'' = -\lambda x^+ + \mu x^-, \tag{9}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. In this case the function $U(\alpha, \lambda) = \frac{\pi}{\sqrt{\lambda}}$ is independent of α and it is strictly decreasing and strictly convex in λ,

$$\lim_{\lambda \to 0+} U(\alpha, \lambda) = +\infty, \qquad \lim_{\lambda \to +\infty} U(\alpha, \lambda) = 0. \tag{10}$$

In the sequel the conditions will be obtained for the function $U(\alpha, \lambda)$ to have similar properties (monotonicity and convexity) in the case of the problems for equation (1).

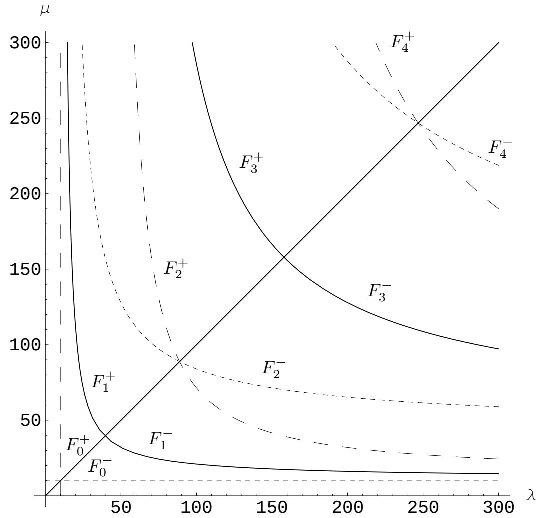

Figure 1: The classical Fučík spectrum for the Dirichlet problem (9), (3), $b - a = 1$.

It is clear that properties of the function $U(\alpha, \lambda)$ are essential for existence and uniqueness (multiplicity) of (positive) solutions to the Dirichlet problem (7), (3): a) if a function $U(\alpha, \lambda)$ is strictly monotone in the first argument for a fixed $\lambda > 0$, then the Dirichlet problem (7), (3) has at most one positive solution, b) if a function $U(\alpha, \lambda)$ is strictly convex (concave) in the first argument for a fixed $\lambda > 0$, then the Dirichlet problem (7), (3) has at most two positive solutions.

Deriving formulae for $U(\alpha, \lambda)$ in the following sections we assume that any involved derivatives exist.

2 Time map formulae

2.1 Definitions and expressions

Consider equation (7) together with the initial conditions (8). The *time map* (for nonlinearity f) is the time needed for a solution (x, y) of the equivalent planar system $x' = y$,

$y' = -\lambda f(x)$ to move from the point $(0, \alpha)$ to the point $(0, -\alpha)$ crossing the half-axis $x' = 0$, $x > 0$ once (at the point $(u, 0)$). Since the energy integral associated to (7), (8) is constant:

$$\frac{x'^2}{2} + \lambda F(x) = const, \tag{11}$$

then

$$const = \lambda F(u) = \frac{\alpha^2}{2},$$

where $F(x) = \int\limits_0^x f(s)ds$. Solving (11) for $\frac{dx}{dt}$ and integrating we get two formulae for the time map.

1.

$$U(\alpha, \lambda) = 2\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} \tag{12}$$

or

$$U(\alpha, \lambda) = \frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right), \quad t_1(\alpha) = 2\int_0^{F^{-1}(\frac{\alpha^2}{2})} \frac{dx}{\sqrt{\alpha^2 - 2F(x)}}, \tag{13}$$

where the time map is expressed as a function $U(\alpha, \lambda)$ depending on the derivative α of the solution x at the point $t = a$ and the parameter λ, but $t_1(\alpha) = U(\alpha, 1)$ is the time map for the parameter $\lambda = 1$.

2.

$$\mathcal{U}(u, \lambda) = \sqrt{\frac{2}{\lambda}} \int_0^u \frac{dx}{\sqrt{F(u) - F(x)}} \tag{14}$$

or

$$\mathcal{U}(u, \lambda) = \frac{1}{\sqrt{\lambda}}\, T_1(u), \quad T_1(u) = \sqrt{2}\int_0^u \frac{dx}{\sqrt{F(u) - F(x)}}, \tag{15}$$

where the time map is expressed as a function $\mathcal{U}(u, \lambda)$ depending on the maximum value u of the positive solution x, $x(a) = 0$, $x\,(a + \mathcal{U}(u, \lambda)) = 0$ and the parameter λ, but $T_1(u) = \mathcal{U}(\alpha, 1)$ is the time map for the parameter $\lambda = 1$.

Observe that

$$U\left(\sqrt{2\lambda F(u)}, \lambda\right) = \mathcal{U}(u, \lambda), \quad \mathcal{U}\left(F^{-1}\left(\frac{\alpha^2}{2\lambda}\right), \lambda\right) = U(\alpha, \lambda), \tag{16}$$

$$t_1\left(\sqrt{2F(u)}\right) = T_1(u), \quad T_1\left(F^{-1}\left(\frac{\alpha^2}{2}\right)\right) = t_1(\alpha). \tag{17}$$

Alternative expressions for the time map can be obtained using the technique in the paper [7]. The region in the (x, y)-plane is considered which is bounded by the integral curve $\frac{y^2}{2} + F(x) = h$ of the system $x' = y$, $y' = -f(x)$ and the straight line $x = 0$. The area of this region is given by

$$J_0^+(h) = 2\int\limits_0^{x_+} \sqrt{2(h - F(\xi)}\, d\xi, \quad x_+ = F^{-1}(h) > 0.$$

The time map is represented as a function $T^+(h)$, which depends on the energy level h, and $T^+(h) = \frac{d}{dh}\, J_0^+(h)$.

The new variable s is introduced as $F(\xi) = sh$. Differentiating, $d\xi = \frac{hds}{f(\xi)}$. Introducing $\eta := \sqrt{2(h - F(\xi))} = \sqrt{2(1-s)h}$, one gets

$$J_0^+(h) = 2\int_0^1 \frac{h\eta}{f(\xi)}\, d\xi, \quad T^+(h) = 2\int_0^1 \left(\frac{h\eta}{f(\xi)}\right)_h' \, d\xi.$$

Then:

$$\left(\frac{h\eta}{f(\xi)}\right)_h' = \frac{\eta f(\xi) + h\eta_h' f(\xi) - \eta f'(\xi) h \frac{d\xi}{dh}}{f^2(\xi)};$$

it follows from $F(\xi) = sh$ that $F'(\xi)\frac{d\xi}{dh} = s$, therefore $\frac{d\xi}{dh} = \frac{s}{f(\xi)}$ or $h\frac{d\xi}{dh} = \frac{hs}{f(\xi)} = \frac{F(\xi)}{f(\xi)}$;

$$h\eta_h' = h\left(\sqrt{2(h - F(\xi))}\right)_h' = \frac{h(2(h-F(\xi)))_h'}{2\eta} = \frac{2h - 2hF'(\xi)\frac{d\xi}{dh}}{2\eta} =$$

$$= \frac{2h - 2f(\xi)\frac{F(\xi)}{f(\xi)}}{2\eta} = \frac{2h - 2F(\xi)}{2\eta} = \frac{\eta^2}{2\eta} = \frac{\eta}{2};$$

$$\left(\frac{h\eta}{f(\xi)}\right)_h' = \frac{\eta f(\xi) + \frac{\eta}{2} f(\xi) - \eta f'(\xi)\frac{F(\xi)}{f(\xi)}}{f^2(\xi)} = \left(\frac{3}{2} - \frac{F(\xi)F''(\xi)}{f^2(\xi)}\right)\frac{\eta}{f(\xi)}.$$

Hence

$$T^+(h) = 2\int_0^1 \left(\frac{3}{2} - \frac{F(\xi)F''(\xi)}{f^2(\xi)}\right)\eta\,\frac{ds}{f(\xi)} = \frac{2}{h}\int_0^1 \left(\frac{3}{2} - \frac{F(\xi)F''(\xi)}{f^2(\xi)}\right)\eta\,\frac{hds}{f(\xi)} =$$

$$= \frac{2}{h}\int_0^{x_+} \left(\frac{3}{2} - \frac{F(\xi)F''(\xi)}{f^2(\xi)}\right)\sqrt{2(h - F(\xi))}d\xi.$$

3. Since $h = \frac{\alpha^2}{2}$,

$$U(\alpha, \lambda) = \frac{1}{\sqrt{\lambda}}\, t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) = \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{\alpha^2 - 2\lambda F(x)}dx, \tag{18}$$

$$t_1(\alpha) = T^+\left(\frac{\alpha^2}{2}\right) = \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2})} \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{\alpha^2 - 2F(x)}dx. \tag{19}$$

4. One gets from (18) and (19), making use of (17), that

$$\mathcal{U}(u, \lambda) = \frac{1}{\sqrt{\lambda}}\, T_1(u) = \frac{2^{\frac{3}{2}}}{\sqrt{\lambda}\, F(u)}\int_0^u \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{F(u) - F(x)}dx, \tag{20}$$

$$T_1(u) = t_1\left(\sqrt{2F(u)}\right) = \frac{2^{\frac{3}{2}}}{F(u)}\int_0^u \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{F(u) - F(x)}dx. \tag{21}$$

Remark 2.1. Acting like Bin-Liu in the paper [7] when obtaining the formula

$$T^+(h) = \frac{2}{h}\int_0^{F^{-1}(h)} \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{2(h - F(x))}dx, \tag{22}$$

we can derive the following expressions

$$\frac{d}{dh}T^+(h) = \frac{2}{h^2}\int_0^{F^{-1}(h)} \left(\frac{3}{4} - 3\frac{F(x)F''(x)}{f^2(x)} + 3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - \frac{F^2(x)F'''(x)}{f^3(x)}\right)\sqrt{2(h - F(x))}dx, \tag{23}$$

$$\frac{d^2}{dh^2}T^+(h)=\frac{2}{h^3}\int_0^{F^{-1}(h)}\left[-\frac{3}{8}-\frac{9}{4}\frac{F(x)F''(x)}{f^2(x)}+\frac{27}{2}\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-15\left(\frac{F(x)F''(x)}{f^2(x)}\right)^3-\right.$$
$$\left.-\frac{9}{2}\frac{F^2(x)F'''(x)}{f^3(x)}-\frac{F^3(x)F^{IY}(x)}{f^4(x)}+10\frac{F^3(x)F''(x)F'''(x)}{f^5(x)}\right]\sqrt{2(h-F(x))}dx. \quad (24)$$

Remark 2.2. In the paper [7] the formula for derivative of

$$T^+(h)=2\int_0^{F^{-1}(h)}\frac{dx}{\sqrt{2(h-F(x))}} \quad (25)$$

was obtained, namely,

$$\frac{d}{dh}T^+(h)=\frac{2}{h}\int_0^{F^{-1}(h)}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{2(h-F(x))}}. \quad (26)$$

The following is valid also:

$$\frac{d^2}{dh^2}T^+(h)=\frac{2}{h^2}\int_0^{F^{-1}(h)}\left(-\frac{1}{4}-\frac{F(x)F''(x)}{f^2(x)}-\frac{F^2(x)F'''(x)}{f^3(x)}+3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2\right)\frac{dx}{\sqrt{2(h-F(x))}}. \quad (27)$$

2.2 Formulae for derivatives - the first series

In this section we calculate the first and second order derivatives of the function $U(\alpha,\lambda)$, which is given by (12).

2.2.1 Calculation of $t_1'(\alpha)$, $t_1''(\alpha)$

It follows from $t_1(\alpha)=T^+\left(\frac{\alpha^2}{2}\right)$ and (26) that

$$t_1'(\alpha)=\left(\frac{\alpha^2}{2}\right)'_\alpha\cdot\left.\frac{dT^+(h)}{dh}\right|_{h=\frac{\alpha^2}{2}}=\alpha\frac{2}{\frac{\alpha^2}{2}}\int_0^{F^{-1}(\frac{\alpha^2}{2})}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{2(\frac{\alpha^2}{2}-F(x))}}=$$
$$=\frac{4}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2})}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2F(x)}},$$

that is,

$$t_1'(\alpha)=\frac{4}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2})}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2F(x)}}. \quad (28)$$

One gets, differentiating $t_1'(\alpha) = \alpha \left.\frac{dT^+(h)}{dh}\right|_{h=\frac{\alpha^2}{2}}$ and making use of (26) and (27), that

$$t_1''(\alpha) = \left.\frac{dT^+(h)}{dh}\right|_{h=\frac{\alpha^2}{2}} + \alpha^2 \left.\frac{d^2T^+(h)}{dh^2}\right|_{h=\frac{\alpha^2}{2}} =$$

$$= \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2})} \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2F(x)}} +$$

$$+ \alpha^2 \frac{8}{\alpha^4}\int_0^{F^{-1}(\frac{\alpha^2}{2})} \left(-\frac{1}{4} - \frac{F(x)F''(x)}{f^2(x)} - \frac{F^2(x)F'''(x)}{f^3(x)} + 3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2\right) \frac{dx}{\sqrt{\alpha^2 - 2F(x)}} =$$

$$= \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2})} \left(6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2F(x)}},$$

that is,

$$t_1''(\alpha) = \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2})} \left(6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2F(x)}}. \tag{29}$$

2.2.2 First order derivatives

$$\frac{\partial U}{\partial \alpha}(\alpha,\lambda) = \left[\frac{1}{\sqrt{\lambda}} t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right)\right]'_\alpha = \frac{1}{\lambda} \left.\frac{dt_1(\gamma)}{d\gamma}\right|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}} =$$

$$= \frac{1}{\lambda}\frac{4}{\frac{\alpha}{\sqrt{\lambda}}}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\frac{\alpha^2}{\lambda} - 2F(x)}} =$$

$$= \frac{4}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}},$$

that is,

$$\frac{\partial U}{\partial \alpha}(\alpha,\lambda) = \frac{4}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}}. \tag{30}$$

$$\frac{\partial U}{\partial \lambda}(\alpha,\lambda) = \left[\frac{1}{\sqrt{\lambda}} t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right)\right]'_\lambda = -\frac{1}{2\lambda\sqrt{\lambda}} t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) - \frac{\alpha}{2\lambda^2} \left.\frac{dt_1(\gamma)}{d\gamma}\right|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}} =$$

$$= -\frac{1}{2\lambda\sqrt{\lambda}}\, 2\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \frac{dx}{\sqrt{\frac{\alpha^2}{\lambda} - 2F(x)}} - \frac{\alpha}{2\lambda^2}\frac{4}{\frac{\alpha}{\sqrt{\lambda}}}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\frac{\alpha^2}{\lambda} - 2F(x)}} =$$

$$= -\frac{1}{\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left[1 + 2\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\right] \frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} =$$

$$= -\frac{2}{\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})} \left(1 - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}},$$

and finally,

$$\frac{\partial U}{\partial \lambda}(\alpha,\lambda) = -\frac{2}{\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left(1-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}}. \tag{31}$$

Remark 2.3. It follows from (30) and (31) that for any $\alpha > 0$, $\lambda > 0$ the identity

$$\alpha\,\frac{\partial U}{\partial \alpha}(\alpha,\lambda) + 2\lambda\,\frac{\partial U}{\partial \lambda}(\alpha,\lambda) = -U(\alpha,\lambda) \tag{32}$$

is valid.

2.2.3 Second order derivatives

One obtains from

$$\frac{\partial U}{\partial \lambda}(\alpha,\lambda) = -\frac{1}{2\lambda\sqrt{\lambda}}t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) - \frac{\alpha}{2\lambda^2}\,\frac{dt_1(\gamma)}{d\gamma}\bigg|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}}$$

that

$$\begin{aligned}
\frac{\partial^2 U}{\partial \lambda^2}(\alpha,\lambda) &= \frac{3}{4\lambda^2\sqrt{\lambda}}\,t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) + \frac{5\alpha}{4\lambda^3}\,\frac{dt_1(\gamma)}{d\gamma}\bigg|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}} + \frac{\alpha^2}{4\lambda^3\sqrt{\lambda}}\,\frac{d^2t_1(\gamma)}{d\gamma^2}\bigg|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}} = \\
&= \frac{3}{4\lambda^2\sqrt{\lambda}}\,2\sqrt{\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} + \\
&+ \frac{5\alpha}{4\lambda^3}\,\frac{4\lambda}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} + \\
&+ \frac{\alpha^2}{4\lambda^3\sqrt{\lambda}}\,\frac{4\lambda\sqrt{\lambda}}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[-3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)} + 6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} = \\
&= \frac{2}{\lambda^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 4\frac{F(x)F''(x)}{f^2(x)} + 2 - \frac{F^2(x)F'''(x)}{f^3(x)} + \right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}},
\end{aligned}$$

that is,

$$\frac{\partial^2 U}{\partial \lambda^2}(\alpha,\lambda) = \frac{2}{\lambda^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 4\frac{F(x)F''(x)}{f^2(x)} + 2 - \frac{F^2(x)F'''(x)}{f^3(x)} + \right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{33}$$

$$= \frac{2}{\lambda^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[3\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right) + \frac{3}{4} - \frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}}. \tag{34}$$

It follows from

$$\frac{\partial U}{\partial \alpha}(\alpha,\lambda) = \frac{1}{\lambda}\,\frac{dt_1(\gamma)}{d\gamma}\bigg|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}}$$

that

$$\frac{\partial^2 U}{\partial \alpha^2}(\alpha,\lambda) = \frac{1}{\lambda\sqrt{\lambda}}\ \frac{d^2 t_1(\gamma)}{d\gamma^2}\bigg|_{\gamma=\frac{\alpha}{\sqrt{\lambda}}} =$$

$$= \frac{1}{\lambda\sqrt{\lambda}}\frac{4}{\frac{\alpha^2}{\lambda}}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\frac{\alpha^2}{\lambda} - 2F(x)}} =$$

$$= \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}},$$

that is,

$$\frac{\partial^2 U}{\partial \alpha^2}(\alpha,\lambda) = \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} \tag{35}$$

$$= \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}}. \tag{36}$$

One obtains, making use of (32), that

$$\frac{\partial^2 U}{\partial \lambda\partial \alpha}(\alpha,\lambda) = -\frac{1}{\lambda}\ \frac{\partial^2 U}{\partial \lambda\partial \alpha}(\alpha,\lambda) - \frac{\alpha}{2\lambda}\ \frac{\partial^2 U}{\partial \alpha^2}(\alpha,\lambda) =$$

$$= -\frac{1}{\lambda}\ \frac{4}{\alpha}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} -$$

$$-\frac{\alpha}{2\lambda}\ \frac{4}{\alpha^2}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} =$$

$$= -\frac{2}{\alpha\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 5\frac{F(x)F''(x)}{f^2(x)} + 1 - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}},$$

that is,

$$\frac{\partial^2 U}{\partial \lambda\partial \alpha}(\alpha,\lambda) = -\frac{2}{\alpha\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 5\frac{F(x)F''(x)}{f^2(x)} + 1 - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}} \tag{37}$$

$$= -\frac{2}{\alpha\lambda}\int_0^{F^{-1}(\frac{\alpha^2}{2\lambda})}\left[6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2 - 2\lambda F(x)}}. \tag{38}$$

With help of (32), the relation

$$\frac{\partial^2 U}{\partial \alpha\partial \lambda}(\alpha,\lambda) = -\frac{3}{\alpha}\ \frac{\partial U}{\partial \lambda}(\alpha,\lambda) - \frac{2\lambda}{\alpha}\ \frac{\partial^2 U}{\partial \lambda\partial \lambda}(\alpha,\lambda)$$

is obtained and it follows from (31) and (33) that for any $\alpha > 0$, $\lambda > 0$

$$\frac{\partial^2 U}{\partial \alpha\partial \lambda}(\alpha,\lambda) = \frac{\partial^2 U}{\partial \lambda\partial \alpha}(\alpha,\lambda). \tag{39}$$

2.3 Formulae for derivatives of the time map - the first series

$$U(\alpha,\lambda)=\int_0^u \frac{2dx}{\sqrt{\alpha^2-2\lambda F(x)}},\quad u=F^{-1}\left(\frac{\alpha^2}{2\lambda}\right) \tag{40}$$

$$\frac{\partial U}{\partial \alpha}(\alpha,\lambda)=\frac{4}{\alpha}\int_0^u \left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{41}$$

$$\frac{\partial U}{\partial \lambda}(\alpha,\lambda)=-\frac{2}{\lambda}\int_0^u \left(1-\frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{42}$$

$$\alpha\,\frac{\partial U}{\partial \alpha}(\alpha,\lambda)+2\lambda\,\frac{\partial U}{\partial \lambda}(\alpha,\lambda)=-U(\alpha,\lambda) \tag{43}$$

$$\frac{\partial^2 U}{\partial \alpha^2}(\alpha,\lambda)=\frac{4}{\alpha^2}\int_0^u \left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-3\frac{F(x)F''(x)}{f^2(x)}-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{44}$$

$$=\frac{4}{\alpha^2}\int_0^u \left[6\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2-3\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{45}$$

$$\frac{\partial^2 U}{\partial \lambda^2}(\alpha,\lambda)=\frac{2}{\lambda^2}\int_0^u \left[3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-4\frac{F(x)F''(x)}{f^2(x)}+2-\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{46}$$

$$=\frac{2}{\lambda^2}\int_0^u \left[3\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2+\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)+\frac{3}{4}-\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{47}$$

$$\frac{\partial^2 U}{\partial \alpha\partial \lambda}(\alpha,\lambda)=-\frac{2}{\alpha\lambda}\int_0^u \left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-5\frac{F(x)F''(x)}{f^2(x)}+1-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{48}$$

$$=-\frac{2}{\alpha\lambda}\int_0^u \left[6\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2-\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\frac{dx}{\sqrt{\alpha^2-2\lambda F(x)}} \tag{49}$$

$$=\frac{\partial^2 U}{\partial \lambda\partial \alpha}(\alpha,\lambda)$$

2.4 Formulae for derivatives of the time map - the second series

Arguing like in section 2.2, the first and second order partial derivatives can be found for the function $U(\alpha,\lambda)$ given by (18).

$$U(\alpha,\lambda)=\frac{4}{\alpha^2}\int_0^u\left(\frac{3}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)\sqrt{\alpha^2-2\lambda F(x)}dx,\quad u=F^{-1}\left(\frac{\alpha^2}{2\lambda}\right) \tag{50}$$

$$\frac{\partial U}{\partial\alpha}(\alpha,\lambda)=\frac{8}{\alpha^3}\int_0^u\left[3\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-3\frac{F(x)F''(x)}{f^2(x)}+\frac{3}{4}-\frac{F^2(x)F'''(x)}{f^3(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{51}$$

$$=\frac{8}{\alpha^3}\int_0^u\left[3\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2-\frac{F^2(x)F'''(x)}{f^3(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{52}$$

$$\frac{\partial U}{\partial\lambda}(\alpha,\lambda)=-\frac{2}{\alpha^2\lambda}\int_0^u\left[6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2-7\frac{F(x)F''(x)}{f^2(x)}+3-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{53}$$

$$=-\frac{2}{\alpha^2\lambda}\int_0^u\left[6\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2+\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)+1-\right.$$
$$\left.-2\frac{F^2(x)F'''(x)}{f^3(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{54}$$

$$\alpha\,\frac{\partial U}{\partial\alpha}(\alpha,\lambda)+2\lambda\,\frac{\partial U}{\partial\lambda}(\alpha,\lambda)=-U(\alpha,\lambda) \tag{55}$$

$$\frac{\partial^2 U}{\partial\alpha^2}(\alpha,\lambda)=\frac{16}{\alpha^4}\int_0^u\left[-15\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2\frac{F(x)F''(x)}{f^2(x)}-5\frac{F^2F'''}{f^3}-\right.$$
$$\left.-\frac{F^3F^{IY}}{f^4}+10\frac{F^3(x)F''(x)F'''(x)}{f^5(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{56}$$

$$=-\frac{16}{\alpha^4}\int_0^u\left[15\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2\frac{F(x)F''(x)}{f^2(x)}+10\frac{F^2(x)F'''(x)}{f^3(x)}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)+\right.$$
$$\left.+\frac{F^3(x)F^{IY}(x)}{f^4(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{57}$$

$$\frac{\partial^2 U}{\partial\lambda^2}(\alpha,\lambda)=\frac{1}{\lambda^2\alpha^2}\int_0^u\left[60\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^3+\left(3-40\frac{F^2(x)F'''(x)}{f^3(x)}\right)\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)+\right.$$
$$\left.+3-4\frac{F^3(x)F^{IY}(x)}{f^4(x)}-10\frac{F^2(x)F'''(x)}{f^3(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{58}$$

$$\frac{\partial^2 U}{\partial\alpha\partial\lambda}(\alpha,\lambda)=\frac{\partial^2 U}{\partial\lambda\partial\alpha}(\alpha,\lambda)=-\frac{8}{\lambda\alpha^3}\int_0^u\left[\left(3-15\frac{F(x)F''(x)}{f^2(x)}\right)\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)^2-\right.$$
$$\left.-10\frac{F^2(x)F'''(x)}{f^3(x)}\left(\frac{1}{2}-\frac{F(x)F''(x)}{f^2(x)}\right)-\frac{F^2(x)F'''(x)}{f^3(x)}-\frac{F^3(x)F^{IY}(x)}{f^4(x)}\right]\sqrt{\alpha^2-2\lambda F(x)}dx \tag{59}$$

3 Applications

3.1 Strict monotonicity of a time map in the first argument

Proposition 3.1 *For any fixed $\lambda > 0$ a function $U(\alpha, \lambda)$ is strictly increasing (decreasing) in the first argument, if one of the conditions is fulfilled.*

1. *$f \in C^1(0, +\infty)$ and*

$$\forall x > 0: \; \frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)} > 0 \; (< 0). \tag{60}$$

2. *$f \in C^2(0, +\infty)$ and*

$$\forall x > 0: \; 3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \frac{F^2(x)F'''(x)}{f^3(x)} > 0 \; (< 0). \tag{61}$$

Proof. Follows from (41) and (52).□

Corollary 3.1 *If $f \in C^2(0, +\infty)$ and f is strictly concave, then for any fixed $\lambda > 0$ the function $U(\alpha, \lambda)$ is strictly increasing in the first argument.*

Remark 3.1. It is well known ([9], [4]) that the condition (60) ensures monotonicity of the time map.

Remark 3.2. If one of the conditions (60) or (61) is valid, then the Dirichlet boundary value problem (7), (3) admits at most one solution.

3.2 Strict convexity (concavity) of a time map in the first argument

Proposition 3.2 *For any fixed $\lambda > 0$ a function $U(\alpha, \lambda)$ is strictly convex (concave) in the first argument, if one of the conditions is fulfilled.*

1. *$f \in C^2(0, +\infty)$ and*

$$\forall x > 0: \; 6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 3\frac{F(x)F''(x)}{f^2(x)} - 2\frac{F^2(x)F'''(x)}{f^3(x)} > 0 \; (< 0). \tag{62}$$

2. *$f \in C^3(0, +\infty)$ and*

$$\begin{aligned}\forall x > 0: \; 15\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 \frac{F(x)F''(x)}{f^2(x)} + \\ + 10\frac{F^2(x)F'''(x)}{f^3(x)}\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + \frac{F^3(x)F^{IY}(x)}{f^4(x)} < 0 \; (> 0).\end{aligned} \tag{63}$$

Proof. Follows from (44) and (57).□

Remark 3.3. It is well known ([9], see also [4]) that the condition (62) ensures convexity (concavity) of the time map.

Remark 3.4. If one of the conditions (62) or (63) is valid, then the Dirichlet boundary value problem (7), (3) admits at most two solutions.

Proposition 3.3 *For any fixed $\alpha > 0$ the function $U(\alpha, \lambda)$ is strictly concave (convex) and strictly increasing (decreasing) in the first argument, if one of the conditions*

1. $f \in C^2(0, +\infty)$ *and*

$$\forall x > 0 :\ 6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - 2\frac{F^2(x)F'''(x)}{f^3(x)} < 0(> 0), \tag{64}$$

2. $f \in C^3(0, +\infty)$ *and*

$$\forall x > 0 :\ \left(3 - 15\frac{F(x)F''(x)}{f^2(x)}\right)\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \\ - 10\frac{F^2(x)F'''(x)}{f^3(x)}\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - \frac{F^2(x)F'''(x)}{f^3(x)} - \frac{F^3(x)F^{IY}(x)}{f^4(x)} < 0(> 0) \tag{65}$$

and one of conditions of Proposition 3.1 are fulfilled.

Proof. One gets differentiating both sides of the identity (43) with respect to α, that for any $\alpha > 0$, $\lambda > 0$ the relation

$$\frac{\partial^2 U}{\partial \alpha^2}(\alpha, \lambda) = -\frac{2\lambda}{\alpha}\,\frac{\partial^2 U}{\partial \lambda \partial \alpha}(\alpha, \lambda) - \frac{2}{\alpha}\,\frac{\partial U}{\partial \alpha}(\alpha, \lambda) \tag{66}$$

holds. Therefore, if $\frac{\partial^2 U}{\partial \lambda \partial \alpha} > 0(< 0)$ and $\frac{\partial U}{\partial \alpha} > 0(< 0)$, then $\frac{\partial^2 U}{\partial \alpha^2} < 0(> 0)$. The proof is finalized by application of (49), (59) and Proposition 3.1.□

Remark 3.5. Proposition 3.3 indicates those functions $U(\alpha, \lambda)$ which are strictly concave (convex) in the first argument of all functions $U(\alpha, \lambda)$ which are strictly increasing (decreasing) in the first argument.

3.3 Strict decrease of a time map in the second argument

Proposition 3.4 *For any fixed $\alpha > 0$ the function $U(\alpha, \lambda)$ is strictly decreasing in the first argument, if one of conditions is fulfilled.*

1. $f \in C^1(0, +\infty)$ *and*

$$\forall x > 0 :\ 1 - \frac{F(x)F''(x)}{f^2(x)} > 0. \tag{67}$$

2. $f \in C^2(0, +\infty)$ *and*

$$\forall x > 0 :\ 6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + 1 - 2\frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{68}$$

3. $f \in C^2(0,+\infty)$ *and*

$$\forall x > 0: \ \frac{23}{48} - \frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{69}$$

4. $f \in C^2(0,+\infty)$ *and*

$$\forall x > 0: \ f''(x) < 0. \tag{70}$$

5.

$$\forall x > 0: \ 3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{71}$$

Proof. 1. un 2. follows from (42) and (54).
3. Observe

$$\forall x > 0: \ A(x) := 6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + 1 \geq \frac{23}{24}.$$

If (69) is true, $A(x) \geq \frac{23}{24} > 2\frac{F^2(x)F'''(x)}{f^3(x)}$, hence (68) is valid.
4. follows from 3.
5. follows from the identity (43) un Proposition 3.1.□

Remark 3.6. For motivation, why only strict decrease of $U(\alpha, \lambda)$ with respect to λ is considered, see Introduction.

Remark 3.7. It follows from (43), that any condition which ensures that the time map $U(\alpha, \lambda)$ is strictly increasing in α for fixed λ, ensures also that $U(\alpha, \lambda)$ is strictly decreasing in λ for fixed α. The condition (60) of Proposition 3.1 does not provide new information, because: if (60) holds, then (67) also holds.

Remark 3.8. The following example shows that the conditions (68) and (71) are different, while similar: if, for example, $f(x) = x^3$, then

$$6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + 1 - 2\frac{F^2(x)F'''(x)}{f^3(x)} = \frac{3}{8} > 0,$$

$$3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \frac{F^2(x)F'''(x)}{f^3(x)} = -\frac{3}{8} < 0.$$

3.4 Strict convexity of a time map in the second argument

Proposition 3.5 *For any fixed $\alpha > 0$ a function $U(\alpha, \lambda)$ is strictly convex in the second argument, if one of the conditions is fulfilled.*

1. $f \in C^2(0,+\infty)$ *and*

$$\forall x > 0: \ 3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + \frac{3}{4} - \frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{72}$$

2. $f \in C^2(0, +\infty)$ *and*

$$\forall x > 0: \ \frac{2}{3} - \frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{73}$$

3. $f \in C^2(0, +\infty)$ *and*

$$\forall x > 0: \ f''(x) < 0. \tag{74}$$

4. $f \in C^3(0, +\infty)$ *and*

$$\forall x > 0: \ 60\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^3 + \left(3 - 40\frac{F^2(x)F'''(x)}{f^3(x)}\right)\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + \\ + 3 - 4\frac{F^3(x)F^{IY}(x)}{f^4(x)} - 10\frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{75}$$

Proof. 1. and 4. follow from (47) and (58).
2. Observe

$$\forall x > 0: \ A(x) := 3\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 + \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) + \frac{3}{4} \geq \frac{2}{3}.$$

If (73) is true, $A(x) \geq \frac{2}{3} > \frac{F^2(x)F'''(x)}{f^3(x)}$, therefore (72) is valid.
3. follows from 2.□

Proposition 3.6 *For any fixed* $\alpha > 0$ *the function* $U(\alpha, \lambda)$ *is strictly convex and strictly decreasing in the second argument, if one of the conditions*

1. $f \in C^2(0, +\infty)$ *and*

$$\forall x > 0: \ 6\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - 2\frac{F^2(x)F'''(x)}{f^3(x)} > 0. \tag{76}$$

2. $f \in C^3(0, +\infty)$ *and*

$$\forall x > 0: \ \left(3 - 15\frac{F(x)F''(x)}{f^2(x)}\right)\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)^2 - \\ - 10\frac{F^2(x)F'''(x)}{f^3(x)}\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) - \frac{F^2(x)F'''(x)}{f^3(x)} - \frac{F^3(x)F^{IY}(x)}{f^4(x)} > 0. \tag{77}$$

and one of the conditions of Proposition 3.3 are fulfilled.

Proof. One gets differentiating both sides of the identity (43) with respect to λ, that for any $\alpha > 0$, $\lambda > 0$ the relation

$$\frac{\partial^2 U}{\partial \lambda^2}(\alpha, \lambda) = -\frac{\alpha}{2\lambda}\frac{\partial^2 U}{\partial \alpha \partial \lambda}(\alpha, \lambda) - \frac{3}{2\lambda}\frac{\partial U}{\partial \lambda}(\alpha, \lambda), \tag{78}$$

holds. Therefore, if $\frac{\partial^2 U}{\partial \alpha \partial \lambda} < 0$ and $\frac{\partial U}{\partial \lambda} < 0$, then $\frac{\partial^2 U}{\partial \lambda^2} > 0$. The proof is completed taking into account (49), (59) and Proposition 3.3.□

Remark 3.9. Proposition 3.6 indicates those functions $U(\alpha,\lambda)$ which are strictly convex in the second argument of all functions $U(\alpha,\lambda)$ which are strictly decreasing in the second argument.

Remark 3.10. It was mentioned in Introduction that the function $U(\alpha,\lambda) = \frac{\pi}{\sqrt{\lambda}}$ associated with the classical Fučík spectrum is strictly convex and satisfies (10). Notice that (10) is valid for any function f, which satisfies

$$0 < \lim_{x\to 0+} \frac{f(x)}{x} = f'(0+) \le +\infty, \quad 0 \le \lim_{x\to +\infty} \frac{f(x)}{x} = f'(+\infty) < +\infty.$$

Indeed, due to [8]

$$0 \le \lim_{u\to 0+} T_1(u) = \frac{\pi}{\sqrt{f'(0+)}} < +\infty, \quad 0 < \lim_{u\to +\infty} T_1(u) = \frac{\pi}{\sqrt{f'(0+)}} \le +\infty. \tag{79}$$

If α is fixed, then $t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right) = T_1(u)$, where $u = F^{-1}\left(\frac{\alpha^2}{2\lambda}\right) \to +\infty$, as $\lambda \to 0+$, and $u \to 0+$, if $\lambda \to +\infty$. (It is preassumed that $\lim\limits_{x\to+\infty} F(x) = +\infty$). Therefore

$$\lim_{\lambda\to 0+} U(\alpha,\lambda) = \lim_{\lambda\to 0+} \frac{1}{\sqrt{\lambda}} \lim_{u\to+\infty} T_1(u) = +\infty, \quad \lim_{\lambda\to+\infty} U(\alpha,\lambda) = \lim_{\lambda\to+\infty} \frac{1}{\sqrt{\lambda}} \lim_{u\to 0+} T_1(u) = 0.$$

The conditions (79) are fulfilled, for instance, if $\frac{f(x)}{x}$ is strictly decreasing function in the interval $(0,+\infty)$ ([6]).

The following assertion is motivated now.

Proposition 3.7 *If* $\lim\limits_{\lambda\to+\infty} U(\alpha,\lambda) = 0$ *and at least one of the conditions of Proposition 3.5 holds, then for any fixed* $\alpha > 0$ *the function* $U(\alpha,\lambda)$ *is strictly decreasing in the second argument.*

Proof. Follows from the properties of a convex function and definition of a limit at infinity.□

The next statement extends the class of functions f, which satisfy the condition (10).

Proposition 3.8 *1. If*

$$f(0) = f'(0+) = \cdots = f^{(k-1)}(0+) = 0, \quad 0 < f^{(k)}(0+) \le +\infty, \tag{80}$$

then

$$\lim_{\lambda\to+\infty} U(\alpha,\lambda) = 0;$$

2. If

$$f(+\infty) = f'(+\infty) = \cdots = f^{(k-1)}(+\infty) = +\infty, \quad 0 \le f^{(k)}(+\infty) < \infty, \tag{81}$$

then

$$\lim_{\lambda\to 0+} U(\alpha,\lambda) = +\infty.$$

Proof. To begin, let us mention that the following assertions can be proved, using [8, Theorem 9]:

a) if

$$\lim_{x\to 0+}\frac{f(x)}{x^k}=\beta\ (0\le\beta\le+\infty),$$

then

$$\lim_{u\to 0+}T(u)u^{\frac{k-1}{2}}=\frac{\sqrt{2}(k+1)^{\frac{1}{2}}A_k}{\sqrt{\beta}};\tag{82}$$

b) if

$$\lim_{u\to +\infty}\frac{f(x)}{x^k}=\beta\ (0\le\beta\le+\infty),$$

then

$$\lim_{u\to +\infty}T(u)u^{\frac{k-1}{2}}=\frac{\sqrt{2}(k+1)^{\frac{1}{2}}A_k}{\sqrt{\beta}},\tag{83}$$

$k=1,2,\ldots,$ $A_k=\int_0^1\frac{ds}{\sqrt{1-s^{k+1}}}$.

1. One has for fixed $\alpha>0$ that

$$\lim_{\lambda\to+\infty}U(\alpha,\lambda)=\lim_{u\to 0+}\frac{\sqrt{2}}{\alpha}\sqrt{F(u)}\,T(u)=$$

$$=\lim_{u\to 0+}\frac{\sqrt{2}}{\alpha}\sqrt{\frac{F(u)}{u^{k-1}}}\,T(u)u^{\frac{k-1}{2}}=\frac{\sqrt{2}}{\alpha}\sqrt{\lim_{u\to 0+}\frac{F(u)}{u^{k-1}}}\,\lim_{u\to 0+}T(u)u^{\frac{k-1}{2}}.$$

Obtain, taking into account (80) and repeatedly applying the L'Hospitale's rule, that

$$\lim_{u\to 0+}\frac{F(u)}{u^{k-1}}=\left|\frac{0}{0}\right|=\lim_{u\to 0+}\frac{f(u)}{(k-1)u^{k-2}}=\left|\frac{0}{0}\right|=\cdots=$$

$$=\left\{\begin{array}{cc}F(u), & k=1,\\ \lim\limits_{u\to 0+}\frac{f^{(k-2)}(u)}{(k-1)!}, & k\ge 2\end{array}\right\}=\left\{\begin{array}{cc}F(0), & k=1,\\ \frac{f^{(k-2)}(0)}{(k-1)!}, & k\ge 2\end{array}\right\}=0.$$

In view of (80),

$$\lim_{x\to 0+}\frac{f(x)}{x^k}=\frac{f^{(k)}(0+)}{k!}=\beta,\quad 0<\beta\le+\infty,$$

therefore in accordance with (82)

$$\lim_{u\to 0+}T(u)u^{\frac{k-1}{2}}=\frac{\sqrt{2}(k+1)^{\frac{1}{2}}A_k}{\sqrt{\beta}}=c,\quad 0\le c<+\infty.$$

Finally

$$\lim_{\lambda\to+\infty}U(\alpha,\lambda)=\frac{\sqrt{2}}{\alpha}\sqrt{\lim_{u\to 0+}\frac{F(u)}{u^{k-1}}}\,\lim_{u\to 0+}T(u)u^{\frac{k-1}{2}}=\frac{\sqrt{2}}{\alpha}\cdot\sqrt{0}\cdot c=0.$$

The assertion 2 can be proved similarly, making use of (83).□

4 Examples

Example 4.1. Let us consider the function $f(x)=\sqrt{x}+x^3\in C[0,+\infty)\cap C^\infty(0,+\infty)$, which is strictly increasing in the interval $[0,+\infty)$, strictly concave in the interval $\left(0,24^{-\frac{2}{5}}\right)$ and strictly convex in the interval $\left(24^{-\frac{2}{5}},+\infty\right)$, besides

$$f'(0+)=+\infty,\quad f(+\infty)=f'(+\infty)=f''(+\infty)=+\infty,\quad f'''(+\infty)=6.$$

The relations (10) follow from Proposition 3.8. Computations, using Mathematica, show that the condition (73) holds. Therefore the function $U(\alpha,\lambda)$ is strictly convex for any fixed α. Due to Proposition 3.7 the function $U(\alpha,\lambda)$ is strictly decreasing for any fixed $\alpha>0$.

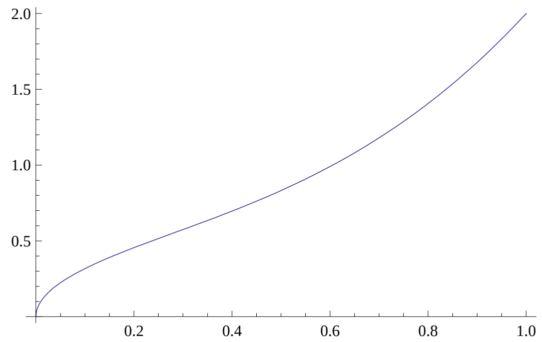

Figure 2: The graph of the function $f(x)=\sqrt{x}+x^3$.

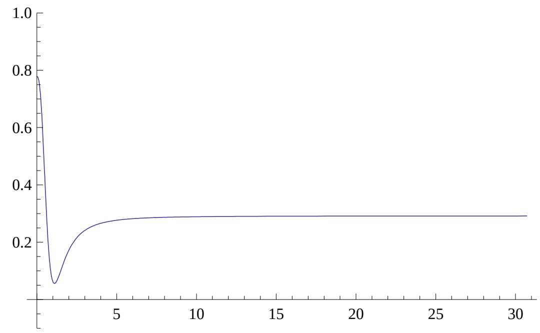

Figure 3: The graph of the function $\varphi(x)=\frac{2}{3}-\frac{F^2(x)F'''(x)}{f^3(x)}$;

$$\lim_{x\to 0+}\varphi(0)=\frac{7}{9},\quad \lim_{x\to+\infty}\varphi(x)=\frac{7}{24},$$
$$x_{min}=1.10127,\quad \varphi(x_{min})=0.0563889.$$

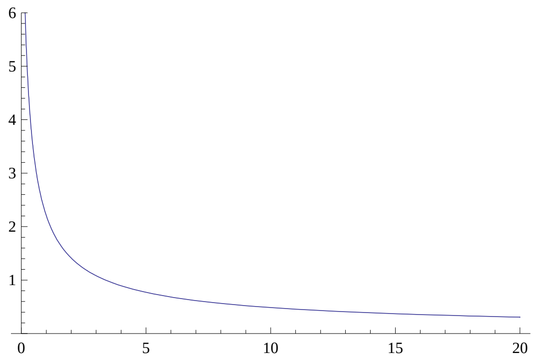

Figure 4: The graph of the function $L=U(0.5,\lambda)$.

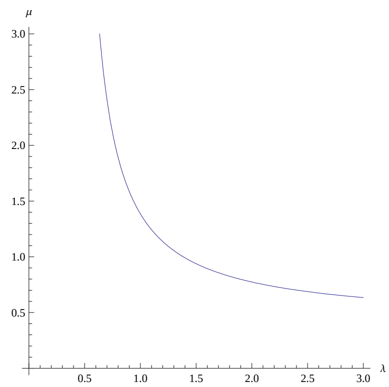

Figure 5: The branch $F_1^{\pm}:\ U(0.5,\lambda)+U(0.5,\mu)=4$ of nonlinear Fučík type spectra for the Dirichlet problem, when $f=g$, on an interval of the length 4.

Example 4.2. Consider the function $f(x)=\sqrt{x}+x^{15}$. Then $f\in C[0,+\infty)\cap C^{\infty}(0,+\infty)$, f is strictly increasing in the interval $[0,+\infty)$, strictly concave in the interval $\left(0,840^{-\frac{2}{29}}\right)$ and strictly convex in the interval $\left(840^{-\frac{2}{29}},+\infty\right)$, besides

$$f'(0+)=+\infty,\quad f(+\infty)=f'(+\infty)=\cdots=f^{(14)}(+\infty)=+\infty,\quad f^{(15)}(+\infty)=15!.$$

The relations (10) follow from Proposition 3.8. Computations, using `Mathematica`, show, see Fig. 7, that the function $U(0.2,\lambda)$ satisfies the conditions of [1, Theorem 2][1]. Therefore

[1] In [1] the normalization condition $\alpha=1$ is supposed, but this is not essential.

we can predict that the branch $F_1^{\pm}$ of nonlinear Fučík type spectra for the Dirichlet problem on the interval $[a, b]$ with the length $b - a = 30$ and the normalization condition $\alpha = 0.2$ consists of two disjoint sets (components), one of them is bounded. This prediction is confirmed by numerical computations, see Fig. 8.

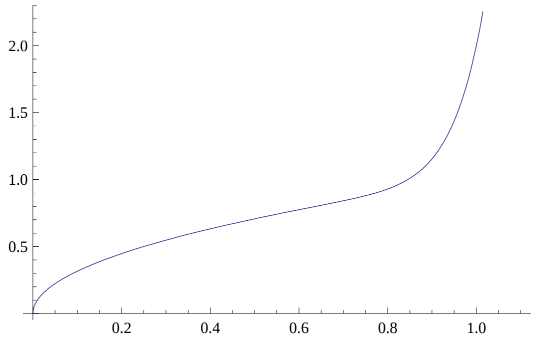

Figure 6: The graph of the function $f(x) = \sqrt{x} + x^{15}$.

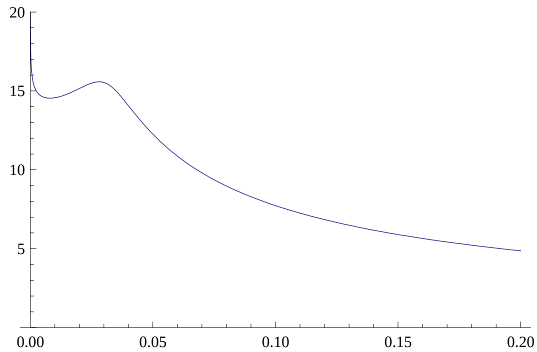

Figure 7: The graph of the function $L = U(0.2, \lambda)$ intersects with the line $L = 15$ at three points with abscissas λ_1, λ_2, λ_3.

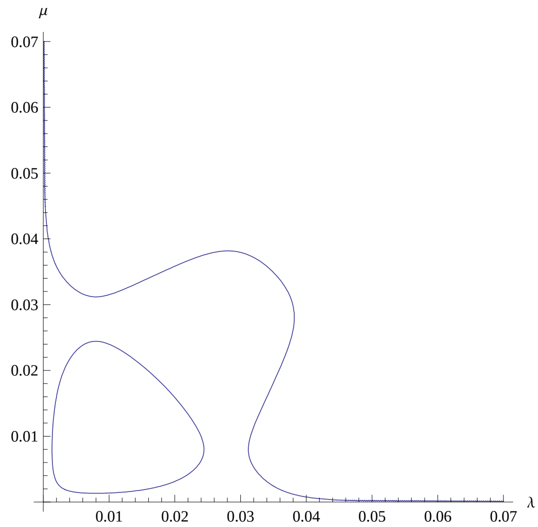

Figure 8: The branch $F_1^{\pm}:\ U(0.2, \lambda) + U(0.2, \mu) = 30$ of nonlinear Fučík type spectra for the Dirichlet problem, when $f = g$, on an interval with the length $30 = 2L$ intersects the bisectrix $\lambda = \mu$ at the points (λ_1, λ_1), (λ_2, λ_2), (λ_3, λ_3).

5 Numerical implementation

Software `Mathematica 6` allows to compute functions $U(\alpha, \lambda)$ and $\mathcal{U}(u, \lambda)$.

- Function f, see Example 4.2.:

$$f[\mathrm{x\_}] := \mathrm{Sign}[x]\sqrt{\mathrm{Abs}[x]} + x^{15}$$

- Function $t_1(\alpha)$, see the formula (13):

$$\text{firstzero}[\alpha\_?\text{NumberQ}]:=\text{Module}[\{p=0\},\text{NDSolve}[\{x"[z]+f[x[z]]==0,x[0]==0,$$
$$x'[0]==\alpha\},x,\{z,\infty\},\text{Method}\to\{\text{EventLocator},\text{Event}\to x[z],$$
$$\text{EventAction}:\to\text{Throw}[p=z,\text{StopIntegration}]\}];p];$$

- Function $U(\alpha,\lambda)$, see the formula (12):

$$U[\alpha\_,\lambda\_]:=\frac{1}{\sqrt{\lambda}}\text{firstzero}\left[\frac{\alpha}{\sqrt{\lambda}}\right];$$

- Function $T_1(u)$, see the formula (15):

$$\text{firstzero1}[\text{u}\_?\text{NumberQ}]:=\text{Module}[\{p=0\},\text{NDSolve}[\{x"[z]+f[x[z]]==0,x[0]==u,$$
$$x'[0]==0\},x,\{z,\infty\},\text{Method}\to\{\text{EventLocator},\text{Event}\to x[z],$$
$$\text{EventAction}:\to\text{Throw}[p=2z,\text{StopIntegration}]\}];p];$$

- Function $\mathcal{U}(u,\lambda)$, see the formula (14):

$$\mathcal{U}[\text{u}\_,\lambda\_]:=\frac{1}{\sqrt{\lambda}}\text{firstzero1}[u];$$

- The graph of the function $U(0.2,\lambda)$, see the Fig. 7:

$$\text{Plot}[\{U[0.2,\lambda]\},\{\lambda,0.000001,0.2\},\text{PlotRange}\to\{0,20\}]$$

- The branch $F_1^{\pm}$, see the Fig. 8:

$$\text{ContourPlot}[\{U[0.2,\lambda]+U[0.2,\mu]==30\},\{\lambda,0.000001,0.07\},\{\mu,0.000001,0.07\},$$
$$\text{PlotPoints}\to 30,\text{Axes}\to\text{True},\text{Frame}\to\text{False},\text{AxesLabel}\to\{\lambda,\mu\}]$$

For EventLocator Method for NDSolve, implemented in `Mathematica 6`, see [10].

6 Conclusions

1. It makes sense to consider the time map as a function of two arguments, which depends on a buit-in in equation parameter λ and another parameter (α or u), which uniquely defines a positive solution of the Dirichlet problem.

2. The obtained time map formulae are useful both for study of multiple solutions of the Dirichlet (Neumann) boundary value problems, and for investigation of nonlinear Fučík type spectra for the Dirichlet (Neumann) BVPs, including structure of the branches.

3. It was observed that considering symmetric ($f=g$) Fučík type equations (1) even for f of the type "strictly concave - strictly convex" the structure of branches of the Dirichlet (Neumann) spectra may differ significantly: for example, the branch $F_1^{\pm}$, which relate to solutions with exactly one zero in (a,b), can be similar to that

of the classical Fučík spectrum and it can consist of two components, one of them bounded.

It is possible that this difference depends on different rates of growth of f in certain intervals.

References

1. A. Gritsans, F. Sadyrbaev. On nonlinear Fučík type spectra, *Math. Model. Anal.*, **13**(2), pp. 203-210, 2008.

2. A. Gritsans, F. Sadyrbaev. Nonlinear spectra: the Neumann problem, Abstracts of MMA2008 & AMOE2008, June 4-7, 2008, Tartu (Kääriku), Estonia, p. 36.

3. A. Gritsans, F. Sadyrbaev. Nonlinear spectra for parameter dependent ordinary differential equations, *Nonlinear Anal., Model. Control*, **12**(2), pp. 253-267, 2007.

4. P. Korman. Global solution branches and exact multiplicity of solutions for two point boundary value problems. In: Canada, A., Drabek, P., Fonda, A. (Eds.), *Handbook of Differential Equations, Ordinary Differential Equations*, Vol. 3. Elsevier Science, North-Holland, Amsterdam, pp. 547-606, 2006.

5. A. Kufner, S. Fučík. *Nonlinear Differential Equations*, Elsevier, Amsterdam-Oxford-New York, 1980.

6. T. Laetsch. The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.*, **20**(1), pp. 1-13, 1970.

7. B. Liu. On Littlewood's boundedness problem for sublinear Duffing equations, *Trans. Am. Math. Soc.*, **353**(4), pp. 1567-1585, 2000.

8. Z. Opial. Sur les périodes des solutions de l'équation différentielle $x'' + g(x) = 0$, *Ann. Polon. Math.*, **10**, pp. 49-72, 1961.

9. R. Schaaf. Global Solution Branches of Two Point Boundary Value Problems, *Lecture Notes in Math.*, Vol. 1458, Springer, 1990.

10. `http://reference.wolfram.com/mathematica/tutorial/NDSolveEventLocator.html`

УДК 517.927

А. Грицанс, Ф. Садырбаев. Различные формулы для функции первых нулей решений автономного обыкновенного дифференциального второго порядка и их приложения.

Аннотация. Рассматриваются различные выражения для функции $U(\alpha, \lambda)$ первых нулей решений задачи $x'' + \lambda f(x) = 0$ (i), $x(a) = 0,\ x'(a) = \alpha > 0$ (ii). Получен

ряд условий для монотонности или выпуклости (вогнутости) для функции U по одному из аргументов при фиксированном втором аргументе, которые применяются а) для оценки числа решений задачи Дирихле (i), $x(a) = x(b) = 0$ (iii) и б) для получения условий подобия нелинейных спектров типа Фучика задачи Дирихле (Неймана) класическому спектру Фучика.

A. Gricāns, F. Sadirbajevs. Otrās kārtas autonoma parastā diferenciālvienādojuma atrisinājumu pirmās nulles funkcijas formulas un to lietojumi

Anotācija. Tiek aplūkotas dažādas problēmas $x'' + \lambda f(x) = 0$ (i), $x(a) = 0,\ x'(a) = \alpha > 0$ (ii) atrisinājumu pirmās nulles funkcijas $U(\alpha, \lambda)$ izteiksmes. Tiek iegūti dažādi nosacījumi, kas garantē funkcijas U monotonitāti vai izliektību attiecībā pret vienu no argumentiem pie fiksēta otrā argumenta, kuri tiek izmantoti, a) lai iegūtu dažus Dirihlē problēmas (i), $x(a) = x(b) = 0$ (iii) atrisinājumu skaita novērtējumus un b) lai iegūtu nosacījumus, pie kuriem Dirihlē (Neimana) problēmas nelineārie Fučika tipa spektri būtu līdzīgi klasiskajam Fučika spektram.

Daugavpils University
Department of Natural Sciences
and Mathematics
Daugavpils, Parades str. 1
arminge@inbox.lv
felix@latnet.lv

Received 11.08.2008