Non-Monotone Iterations

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Summary. A two-point boundary value problem $x'' = f(t,x)$, $x(a) = A$, $x(b) = B$ for which the so called upper and lower functions exist, is considered. For the specific case a non-monotonic iterative approximation scheme has been built for one of the solutions, namely, for which the differential equation of variations oscillates.

1 Introduction

We consider a non-linear second-order differential equation

$$x'' = f(t,x) \tag{1}$$

with boundary conditions

$$x(a) = A, \quad x(b) = B. \tag{2}$$

Geometric interpretation of this problem (1)–(2) is as follows: it is necessary to find the integral curve passing through two points with coordinates (a, A) and (b, B).

Assume there exist upper α and lower β functions for the problem (1)–(2). Functions α and β, according to the definition, are such functions that satisfy the following conditions:

$$\begin{array}{l} \alpha \le \beta, \quad \alpha'' \ge f(t,\alpha), \quad \beta'' \le f(t,\beta), \quad \forall t \in [a,b] \\ \alpha(a) \le A \le \beta(a), \quad \alpha(b) \le B \le \beta(b). \end{array} \tag{3}$$

Then the problem (1)–(2) has a solution $x(t)$, such that ([3],[2])

$$\alpha(t) \le x(t) \le \beta(t), \quad \forall t \in [a,b]$$

We suppose that

$$\alpha < \beta, \quad \alpha(a) < A < \beta(a), \quad \alpha(b) < B < \beta(b) \quad t \in [a,b].$$

For the solution of the problem (1)–(2) there exist schemes of constructing the monotone iterations, which are considered by C. De Coster and P.Habets in their works [4], [1].

The objective of this work is to use the scheme of non-monotone iterations for nonlinear second-order boundary value problem considering the particular example. In Section 2 the scheme of monotone iterations is considered. In Section 3 characteristics of solutions of the analyzed problems are given, as well as the L. Jackson and K. Schrader's theorem is considered. In Section 4 and 5 we use the scheme of non-monotone iterations for solution of the problem (1)–(2). An example of application is shown.

2 Monotone iterative methods

Consider a boundary value problem (1)–(2). Let functions α and β satisfy the conditions (3).

Theorem 2.1 *There exist solutions x^* and x_*, where $x^* \geq x_*$, such that sequences $\{\overline{x_n}\}$ and $\{\underline{x_n}\}$ of solutions of equation (1) can be constructed which converge monotonically.*

Proof. Construct two infinite sequences, which are denoted as $\{\overline{x_n}\}$ and $\{\underline{x_n}\}$. Start with constructing the sequence $\{\overline{x_n}\}$, which converges to the solution of the problem (1)-(2) mentioned above.

Determine the points $\overline{A}_1$ and $\overline{B}_1$ in the following way: $\overline{A}_1 = \beta(a)$; $\overline{B}_1 = \beta(b)$. Then, for the equation (1), the following boundary values are obtained: $x(a) = \overline{A}_1$ and $x(b) = \overline{B}_1$. Such a problem can be solved with the help of the theorem of lower and upper solutions [2, Theorem 7.19]. Denote the obtained solution as $\overline{x}_1$. Now it is possible to rename the upper function, i.e. $\beta(t) = x_1$, because x_1 also satisfies the conditions (3).

Then, for the equation (1) the following boundary values are determined: $x(a) = \overline{A}_2$ and $x(b) = \overline{B}_2$. Denote the solution of this boundary problem as $\overline{x}_2$ and so on. The graph of the function $\overline{x}_2(t)$ lies between graphs of $\alpha(t)$ and $\overline{x}_1(t)$. The following inequalities are valid for the function $\overline{x}_2(t)$

$$\alpha \leq \overline{x}_2(t) \leq \overline{x}_1(t) \leq \beta, \quad \forall t \in [a, b].$$

Now take the function $\overline{x}_2(t)$ as the upper function and consider the next approximation. We obtain an infinite sequence of functions $\{\overline{x_n}\}$ continuing this process.

The following dependence is valid for all members of the sequence $\{\overline{x_n}\}$ according to the elaborating algorithm:

$$\overline{x}_1 \geq \overline{x}_2 \geq \overline{x}_3 \cdots \geq \overline{x}_n \geq \cdots$$

Consider now the selection of points $\overline{A}_i$ and $\overline{B}_i$ $(i = 2, 3, 4, \ldots, n, \ldots)$ for boundary conditions $x(a) = \overline{A}_i, \quad x(b) = \overline{B}_i$.

Let write in general:

$$\overline{A}_i = A + \frac{1}{i}(\beta(a) - A),$$

$$\overline{B}_i = B + \frac{1}{i}(\beta(b) - B),$$

where $i = 2, 3, \cdots$.

The choice of points $\overline{A}_i$ and $\overline{B}_i$ done in this way ensures that the following conditions

$$\alpha(a) < \overline{A}_i < \beta(a),$$
$$\alpha(b) < \overline{B}_i < \beta(b)$$

are satisfied. Also the conditions of the theorem of lower and upper solutions [2, Theorem 7.19] are fulfilled.

Thus, the infinite sequence of solutions of the equation (1) with different boundary values is constructed. Moreover all points $\overline{A}_i$ and $\overline{B}_i$ converge to points A and B, which are defined in the condition (2).

Graphs of all solutions $\overline{x}_i$ $(i = 1, 2, 3 \cdots)$ lie between graphs of functions $\alpha(t)$ and $\beta(t)$. Denote this region as $\omega(\alpha, \beta)$.

Let us prove that a subsequence can be chosen from the infinite sequence of solutions, which converges to x^*.

In the right side of the equation (1) the function $f(t, x)$ can be also unbounded within the interval $[a, b]$. That is why we define the function $F(t, x)$ in the following way:

$$F(t, x) = f(t, \delta(\alpha, x, \beta)) + \delta(0, x - \beta, 1) - \delta(0, \alpha - x, 1), \tag{4}$$

where the function

$$\delta(x, y, z) = \begin{cases} z, & y \geq z, \\ y, & x < y < z, \\ x, & y \leq x. \end{cases} \tag{5}$$

The function $F(t, x)$ according to the above-mentioned definition is bounded, it means that $|F(t, x)| < M$, where $(t, x(t)) \in \omega(\alpha, \beta)$ and $n \in \mathbb{N}$.

Moreover, within the interval $[a, b]$, it coincides with the initial function $f(t, x)$. According to Arzela-Ascoli criterium [2], any infinite compact sequence contains a convergent subsequence. In order to do it, compactness of the infinite number of functions $\{\overline{x}_n\}$ within the space C^1 is to be shown, however it means that $\{\overline{x}_n\}$ and $\{\overline{x}'_n\}$ are equicontinuous and equipotentionally continued ones. Thus, show compactness of the sequence $\{\overline{x_n(t)}\}$ of the solution of the problem (1)-(2). First of all, show that the infinite sequence $\{\overline{x_n(t)}\}$ is bounded. All the solutions of the problem (1), are in $\omega(\alpha, \beta)$. That is why,

$$|\overline{x}_n(t)| < \max\{|\beta(t)|, |\alpha(t)|\}$$

Introduce a constant $K = \max\{|\beta(t)|, |\alpha(t)|\}$.
We get that $\forall t \in [a, b], \quad n \in \mathbb{N} \quad |\overline{x}_n(t)| < K$ is bounded.

Now show that the infinite sequence $\{\overline{x'_n(t)}\}$ is uniformly bounded. Show that there exist t_0 such that the following inequality is true:

$$|\overline{x}'_n(t_0)| < \frac{2K}{b - a}. \tag{6}$$

Let us prove it. Assume the opposite, i.e. $\forall t \in [a, b]$. As a result the following inequality is true:

$$\overline{x}'_n(t) \geq \frac{2K}{b - a} \quad \text{or} \quad \overline{x}'_n(t) \geq -\frac{2K}{b - a}. \tag{7}$$

Integrate both parts of the inequalities (7):

$$\begin{gathered} \int_a^t \overline{x}_n'(s)\,ds \geq \tfrac{2K}{b-a}\int_a^t ds, \quad \text{or} \\ \int_a^t \overline{x}_n'(s)\,ds \leq -\tfrac{2K}{b-a}\int_a^t ds. \end{gathered} \tag{8}$$

As a result, get:

$$\begin{gathered} \overline{x}_n(t) - \overline{x}_n(a) \geq \tfrac{2K}{b-a}(t-a), \quad \text{or} \\ \overline{x}_n(t) - \overline{x}_n(a) \leq -\tfrac{2K}{b-a}(t-a). \end{gathered} \tag{9}$$

In equalities (9) at $t = b$, get:

$$\begin{gathered} \overline{x}_n(b) - \overline{x}_n(a) \geq 2K \quad \text{or} \\ \overline{x}_n(b) - \overline{x}_n(a) \leq -2K. \end{gathered} \tag{10}$$

The last two inequalities contradict to the choice of the number K, what denotes that there exists $t_0 \in [a, b]$, for which the inequality (6) is valid.

Thus, it is shown that the infinite sequence $\{\overline{x}_n'(t)\}$ is uniformly bounded. Before to evaluate $\{\overline{x}_n'(t)\}$ in modulus, write

$$\overline{x}_n'(t) = \overline{x}_n'(t_0) + \int_{t_0}^t \overline{x}_n''(s)\,ds, \tag{11}$$

where $a < t_0 < b$. Then,

$$\begin{aligned} |\overline{x}_n'(t)| &= \left|\overline{x}_n'(t_0) + \int_{t_0}^t \overline{x}_n''(s)\,ds\right| \\ &< \tfrac{2K}{b-a} + \left|\int_{t_0}^t F(s, \overline{x}_n(s)\,ds\right| \\ &< \tfrac{2K}{b-a} + M(b-a). \end{aligned} \tag{12}$$

It can be shown now that the sequence is equicontinuous. First of all, show this feature for the infinite number of functions $\{\overline{x}_n(t)\}$.

According to the definition of that equicontinuty $\forall \varepsilon > 0 \quad \exists \delta > 0$, such that as soon as $|t_2 - t_1| < \delta \quad \Rightarrow |\overline{x}_n(t_1) - \overline{x}_n(t_2)| < \varepsilon$.

One has, according to Lagrange's Mean Value Theorem, that

$$\frac{\overline{x}_n(t_1) - \overline{x}_n(t_2)}{t_1 - t_2} = \overline{x}_n'(\theta), \tag{13}$$

where $t_1 < \theta < t_2 \quad \forall t_1, t_2 \in [a, b]$. We can evaluate the modulus of the difference, using (12).

$$|\overline{x}_n(t_1) - \overline{x}_n(t_2)| = \overline{x}_n'(\theta)|t_1 - t_2| < \left(\frac{2K}{b-a} + M(b-a)\right)|t_1 - t_2|. \tag{14}$$

As a result, get the value $\delta > 0$:

$$\delta = \frac{\varepsilon}{\frac{2K}{b-a} + M(b-a)}. \tag{15}$$

Now evaluate the modulus of the difference $|\overline{x}'_n(t_1) - \overline{x}'_n(t_2)|$ and find the corresponding δ. Using Lagrange's Mean Value Theorem, it is possible to state:

$$|\overline{x}'_n(t_1) - \overline{x}'_n(t_2)| = |\overline{x}''_n(\theta)||t_1 - t_2| \tag{16}$$

where $t_1 < \theta < t_2 \quad \forall t_1, t_2 \in [a, b]$.

Using the condition of the problem (1), in last expression change $\overline{x}''_n(\theta)$ to $F(\theta, \overline{x}_n(\theta))$, and then apply the fact that the function F is limited within the interval $[a, b]$:

$$|\overline{x}'_n(t_1) - \overline{x}'_n(t_2)| = |F(\theta, \overline{x}_n(\theta))||t_1 - t_2| < M|t_1 - t_2|. \tag{17}$$

As the result of the evaluation conducted, get that:

$$\delta = \frac{\varepsilon}{M}.$$

This is proved.

It is possible to select subsequence of the sequence $\overline{x}_1, \overline{x}_2, \overline{x}_3, \cdots, \overline{x}_n, \cdots$, which will converge to a solution of the problem (1)-(2)-call it $x^*(t)$.

Similarly, we can build an infinite sequence of solutions $\{\underline{x_n}\}$, which converges to some solution x_* of the equation (1) on the underside. Considering the boundary conditions

$$x(a) = \underline{A}_1 \quad \text{un} \quad x(b) = \underline{B}_1,$$

where

$$\underline{A_1} = \alpha(a), \quad \underline{B_1} = \alpha(b),$$

we obtain a solution $\underline{x}_1$, for which the following inequality is valid

$$\alpha \le \underline{x}_1(t) \le \beta, \quad \forall t \in [a, b].$$

Now specify once again $\alpha = \underline{x}_1(t)$, i.e. will take function $\underline{x}_1(t)$ as the lower function, because it conforms with the definition of the lower function. The following approximated solution $\underline{x}_2(t)$. The graph of the function $\underline{x}_2(t)$ lies between graphes $\beta(t)$ e $\underline{x}_1(t)$. The following inequality is valid for the function $\underline{x}_2(t)$

$$\alpha \le \underline{x}_1(t) \le \underline{x}_2(t) \le \beta, \quad \forall t \in [a, b].$$

Now take the function $\underline{x}_2(t)$ as the lower function and consider the next approximation. Continuing this process obtain an infinite sequence of functions $\{\underline{x_n}\}$. The following dependence is valid for all elements of the sequence $\{\underline{x_n}\}$ according the constructing algorithm

$$\underline{x}_1 \le \underline{x}_2 \le \underline{x}_3 \cdots \le \underline{x}_n \le \cdots$$

Now consider the choice of points $\underline{A}_i$ and $\underline{B}_i$ $(i = 2, 3, \ldots, n, \ldots)$ for boundary conditions $x(a) = \underline{A}_i, \quad x(b) = \underline{B}_i$. Let us write in the general form:

$$\underline{A}_i = A - \frac{1}{i}(A - \alpha(a)),$$

$$\underline{B}_i = B - \frac{1}{i}(B - \alpha(b)),$$

where $i = 2, 3, \ldots$.

We can choose a subsequence from the infinite set of solutions, which converges to a solution of the boundary problem (1)-(2), denote them by $x_*(t)$. The verification is analogous to the case of the set $\{\overline{x_n}\}$.

Out of the constructed infinite sequence, it is possible to select the subsequence converging to a certain solution of the problem (1)-(2) $x_*(t)$.

After the construction, the following inequality will be valid for the solutions $x^*(t)$ and $x_*(t)$

$$x^*(t) \geq x_*(t),$$

because $\forall n, k\ \overline{x_n}(t) \geq \underline{x_k}(t)$. $\square$

One can show monotone iterations schematically in such a way (Fig. 2.1.).

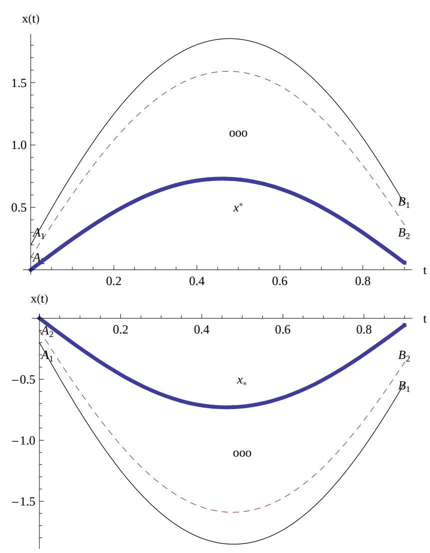

Figure 2.1. The scheme of monotone iterations.

In the work [1, Theorem 2.8], the authors C. De Coster and P. Habets show the mechanism of calculation of analytical entry for the functions of monotone iterations.

Consider the Dirichlet problem

$$u'' = f(t, u), \quad u(a) = 0, \quad u(b) = 0, \tag{18}$$

where f is a continuous function.

Theorem 2.2 *Let α and $\beta \in C^2([a,b])$, $\alpha \le \beta$. Assume $f : \omega(\alpha,\beta) \to \mathbb{R}$ is a continuous function, there exists $M \ge 0$ such that for all $(t,u_1),(t,u_2) \in \omega(\alpha,\beta)$,*

$$u_1 \le u_2 \quad \textit{implies} \quad f(t,u_2) - f(t,u_1) \le M(u_2 - u_1),$$

and for all $t \in [a,b]$

$$\begin{array}{lll} \alpha''(t) \ge f(t,\alpha(t)), & \alpha(a) \le 0, & \alpha(b) \le 0, \\ \beta''(t) \le f(t,\beta(t)), & \beta(a) \ge 0, & \beta(b) \ge 0. \end{array} \tag{19}$$

Then the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by

$$\begin{gathered} \alpha_0 = \alpha \\ \alpha_n'' - M\alpha_n = f(t,\alpha_{n-1}) - M\alpha_{n=1}, \\ \alpha_n(a) = 0, \quad \alpha_n(b) = 0 \end{gathered} \tag{20}$$

and

$$\begin{gathered} \beta_0 = \beta, \\ \beta_n'' - M\beta_n = f(t,\beta n - 1) - M\beta_{n-1} \\ \beta_n(a) = 0, \quad \beta_n(b) = 0 \end{gathered} \tag{21}$$

converge monotonically in $C^1([a,b])$ to solutions u_{min} and u_{max} of (18) such that

$$\alpha \le u_{min} \le u_{max} \le \beta.$$

Further, any solution u of (18) with graph in $\omega(\alpha,\beta)$ verifies

$$u_{min} \le u \le u_{max}.$$

3 Properties of solution

In the article [5] the authors consider the methods of finding the solution for the second-order equations (1) within the given interval, as well as the behaviour of the equation of variations is analysed [5, Theorem 3.3].

Thus, the solution $x(t)$ of (1) on a interval $[a,b]$ has Property B' in case there exists a seguence of solutions x_n such that

- $x_n \to x$ and $x_n' \to x'$ uniformly on $[a,b]$;
- $\triangle_n = x - x_n \ne 0$ and has the same sign for all $n \ge 1$ and $a \le t < b$, or for all $n \ge 1$ and $a < t \le b$;
- for each $0 < \delta < \frac{1}{2}(b-a)$ there is a constant c depending on δ but not on n and t such that $\mid \triangle_n'(t) \mid \le c \mid \triangle_n(t) \mid$ for all $n \ge 1$ and $a + \delta \le t \le b - \delta$.

For simplicity we will assume that $f(t,x)$ is a continuous real-valued function defined on

$$S = \{(t,x) : a \le t \le b, |x| + |x'| < +\infty\}.$$

Theorem 3.1 *Assume that $f(t,x)$ has continuous first order partial derivatives f_x and f'_x with respect to x and x' on S. Let $x_0(t)$ be a solution of $x'' = f(t,x)$ having Property (B') on $[a,b]$. Then the linear equation*

$$y'' = f_x(t, x_0(t))y \tag{22}$$

is disconjugate on the open interval (a,b).

Exploring the differential equation of variations

$$\begin{gathered} y'' = \frac{\partial f}{\partial x}(t, \xi(t))y, \\ y(a) = 0, \quad y'(a) = 1 \end{gathered} \tag{23}$$

for a definite solution $\xi(t)$ of the differential equation (1), we will use the notions of "0-type solution" or "1-type solution" in accordance with definitions in the paper [6].

The authors L. Jackson and K. Schrader in their theorem (Theorem 3.1) mention zero-type solutions.

In our case, the task is to find non-zero-type solutions of the boundary problem(1)-(2), i.e. such solutions that the respective equations of variations (23) are oscillatory within the interval (a,b) and satisfy the remaining characteristics of (B').

4 Non-monotone iterative method

Monotone schemes are likely to be applicable if there are regular $(\alpha < \beta)$ α and β. Iterations converge then to solutions x^* and x_*, which are described in terms of the equations of variations in the following way: the respective equation of variations is disconjugate within the interval $[a,b]$.

As there exist examples of equations, which have α and β, but there are solutions $x(t)$, for which the equation of variations is not disconjugate within the interval (a,b) there appears the need to construct alternative non-monotone iterative schemes.

Let there exist a 1-type solution $x(t)$ of the problem (1)-(2). Then there exists a non-monotone iterative scheme.

Describe mechanism of constructing the non-monotone iterative scheme. As well as in the case of monotone iterations, determine boundary conditions, only in this case take points on different sides of A and B. For example, if $x(a) \geq A$, then $x(b) \leq B$ and vice verso (see Fig. 5.1).

Let us assume at the beginning that $x(a) > A$ and $x(b) < B$, and set the first boundary condition for the equation (1):

$$x(a) = A_1, \quad x(b) = B_1,$$

where

$$\alpha(a) < A_1 < \beta(a), \quad \alpha(b) < B_1 < \beta(b).$$

Accordingly [2, Theorem 7.19], a solution exists for the equation (1) which satisfies these boundary conditions. Denote it u_1. Afterwards define the next boundary conditions for the differential equation (1):

$$x(a) = A_2, \quad x(b) = B_2$$

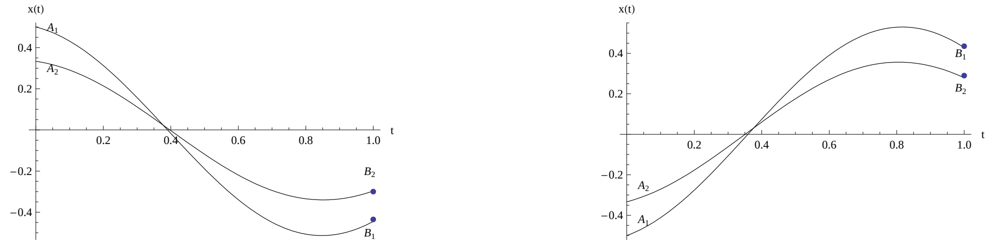

Figure 4.1. Choice of the boundary conditions, if $A = B = 0$.

and derive a solution u_2. The following inequalities are valid

$$\alpha(a) < A_2 < A_1 < \beta(a),$$

$$\alpha(b) < B_1 < B_2 < \beta(b).$$

Now consider the choice of points A_i and B_i $(1 = 2, 3, \ldots, n, \ldots)$ for the boundary conditions $x(a) = A_i, \quad x(b) = B_i$.

Let us write in the general form:

$$A_i = A + \frac{1}{i+1}(\beta(a) - A),$$

$$B_i = B - \frac{1}{i+1}(B - \alpha(b)),$$

where $i = 1, 2, 3, \cdots$.

In the same way continue the algorithm. Take into account that all points A_i, where $i = 1, 2, 3\cdots$, converge to the point A from above, and all points B_i $(i = 1, 2, 3\cdots)$ converge to the point B from below. Continuing this process, an infinite solutions sequence

$$u_1, u_2, \cdots, u_n, \cdots \tag{24}$$

is constructed for the differential equation (1) with different boundary conditions. By Theorem 4.1, one can choose a subsequence from the sequence (24), which converges to a solution of the boundary value problem (1)-(2), denote it u^*. Analogously, construct the infinite sequence of solutions $\{v_n\}$. Let us assume that $x(a) \leq A \quad$ un $\quad x(b) \geq B$.

Due to [2, Theorem 7.19], a solution exists for the equation (1) with the boundary condition

$$x(a) = A_1, \quad x(b) = B_1,$$

where

$$\alpha(a) < A_1 < \beta(a), \quad \alpha(b) < B_1 < \beta(b).$$

Denote this solution v_1. Then define the next boundary conditions for the differential equation (1):

$$x(a) = A_2, \quad x(b) = B_2$$

and obtain the solution v_2. At the same time, the following inequalities have to be valid:

$$\alpha(a) < A_1 < A_2 < \beta(a),$$

$$\alpha(b) < B_2 < B_1 < \beta(b).$$

And so on, continuing the algorithm.

Now consider the choice of points A_i and B_i $(i = 1, 2, 3, \dots, n, \dots)$ for the boundary conditions $x(a) = A_i, \quad x(b) = B_i$.

Let us write in general form:

$$A_i = A - \frac{1}{i+1}(A - \alpha(a)),$$

$$B_i = B + \frac{1}{i+1}(\beta(b) - B),$$

where $i = 1, 2, 3, \cdots$. Take into account that all points A_i $(i = 1, 2, 3 \cdots)$ converge to the point A from above and all points B_i $(i = 1, 2, 3 \cdots)$ converge to the point B from below. We apply Theorem 4.1 to the obtained sequence of solutions

$$v_1, v_2, \cdots, v_n, \cdots \tag{25}$$

and denote a solution, to which the subsequence of the sequence (25) converges, by v_*

In special case, the equality $u^* = v_*$ can be valid for the solutions u^* and v_*.

It follows from the construction that points A_i and B_i converge to points A and B given in the conditions (2).

Theorem 4.1 *From sequences $\{u_n\}$ and $\{v_n\}$ subsequences which converge (may be non monotonically) to a solution of the problem (1)-(2) can be selected.*

Proof. We can show like in the proof of Theorem 2.1 that sequences $\{u_n\}$ and $\{v_n\}$ satisfy the Arzela-Ascoli criterium. □

In the next section, this scheme is applied to construct an example.

5 Example

Let us give an example, in which there exist both upper and lower functions α and β, and show that in addition to solutions mentioned in L.Jackson-K.Schrader's theorem there exist other solutions. As the result of analysis made in terms of the equation of variations, properties of these solutions differ. For L.Jackson-K.Schrader's solutions it is possible to construct monotone iterative schemes, while for the trivial solution such a scheme does not exist. Let us construct non-monotone sequences leading to the trivial solution. So, let us consider the boundary value problem

$$x'' = x^3 - kx \quad \text{where} \quad k = 12 \tag{26}$$

together with the boundary conditions

$$x(0) = x(1) = 0. \tag{27}$$

The upper and lower functions are $\beta = 4$ and $\alpha = -4$, since they satisfy the conditions (3).

Undoubtedly, $\xi \equiv 0$ is the trivial solution of problems (26)–(27). Let us construct the equation of variations for the solution $\xi \equiv 0$ of the problem (26)–(27).

$$y'' = f_x(t, \xi(t)) \mid_{\xi \equiv 0} y = -ky \quad \text{where} \quad k = 12 \tag{28}$$

Also consider the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1. \tag{29}$$

The solution of the problem (28)–(29) is (Figure 5)

$$y = \frac{1}{\sqrt{k}} \sin \sqrt{k}\, t \quad \text{where} \quad k = 12 \tag{30}$$

In our case a solution $y(t)$ of the equation of variations with respect to $\xi \equiv 0$ has the zero in the interval $(0; 1)$. And it follows that $\xi(t)$ is a 1-type solution of the problem.

Figure 5.1. **a.** The solution of the problem (28)-(29);
b. The L.Jackson-K.Schrader's solutions.

According to L. Jackson-K. Schrader's theorem it follows that there exists a solution such that the respective equation (28) is disconjugate in the interval $(0; 1)$ (Figure 5.1 a). The Figure 5.2 gives examples of monotone iterations, with the help of which L. Jackson-K. Schrader's solutions can be approximated.

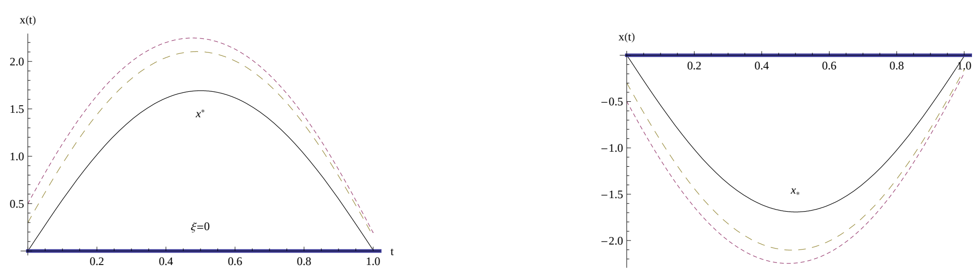

Figure 5.2. The monotone iteration of L. Jackson-K. Schrader's
solutions x^* and x_*.

Let us construct non-monotone sequences converging to the trivial solution. By giving different boundary values (Figure 5.3), according to the scheme described in Section 4, the following sequences of solutions are obtained (Figure 5.4).

Having fixed one of the boundary values $x(0) = 0$, it is possible to notice that all approximations intersect at the point $t = \frac{\pi}{\sqrt{k}}$. It is possible to choose the subsequence of the obtained infinite sequence of solutions, which converges to the trivial solution (Figure 5.5).

Thus the non-monotone iterative scheme is constructed.

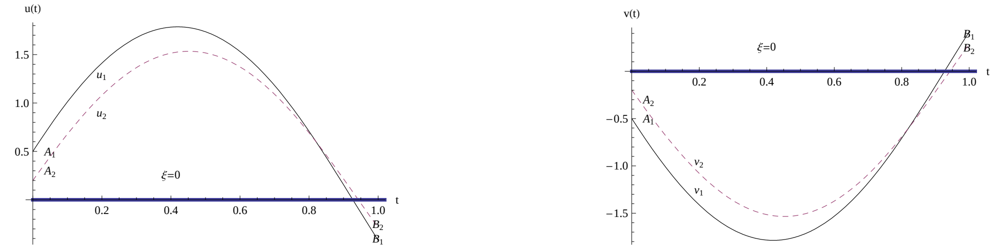

Figure 5.3. The curves of solutions u_1, u_2 and v_1, v_2.

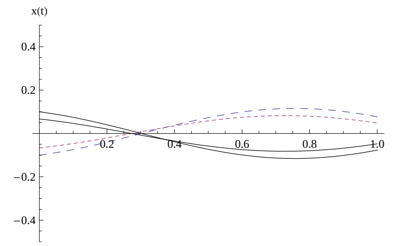

Figure 5.4. Non-monotone approximations of the trivial solution.

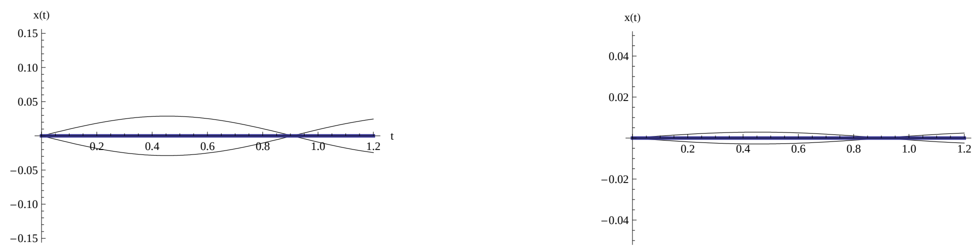

Figure 5.5. Non-monotone approximations for fixed boundary condition $x(0) = 0$.

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М. Добкевич. Немонотонные итеративные схемы.

Аннотация. Рассматриваются двуточечная краевая задача $x'' = f(t,x)$, $x(a) = A$, $x(b) = B$, для которой существуют, так называемые верхняя и нижняя функции. На конкретном примере стоится немонотонная итеративная схема приближения одного из решений задачи, а именно, для которого дифференциальное уравнение в вариациях осцилирует.

УДК 517.927

M. Dobkeviča. Nemonotonas iteratīvas shēmas.

Anotācija. Tiek apskatīts divu punktu robežproblēma $x'' = f(t,x)$, $x(a) = A$, $x(b) = B$ kurai eksistē tā saucamās augšējā un apakšējā funkcijas. Uz noteiktā piemēra vienam no problēmas risinājumiem, respektīvi, tam, kuram diferenciālvienādojums oscilē variācijās, tiek veidota nemonotona itera-tīvā shēma.

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