Comparison of the Dirichlet and Neumann boundary value problems for a certain equation with period annuli

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Summary. We consider the specific second order autonomous equation which has multiple period annuli (connected continua of periodic solutions). These period annuli contain also solutions of the Dirichlet and Neumann problem. We compute them and give theoretical explanation of multiplicity of solutions. Comparison of both problems have been made.

MSC: 34B15, 34C25

1 Introduction

We provide the multiplicity results for the boundary value problem (BVP) where the second order differential equation is of the form

$$x'' + g(x) = 0. \tag{1}$$

From the point of view of the BVP periodic solutions with appropriate periods may satisfy some prescribed boundary conditions. We consider the Dirichlet boundary conditions

$$x(0) = 0, \quad x(1) = 0 \tag{2}$$

and the Neumann boundary conditions

$$x'(0) = 0, \quad x'(1) = 0. \tag{3}$$

Consider the equivalent two-dimensional differential system

$$x' = y, \quad y' = -g(x). \tag{4}$$

It has critical points at $(p_i, 0)$, where p_i are simple zeros of $g(x)$. Recall that a critical point O of (4) is a center if it has a punctured neighborhood covered with nontrivial cycles.

Definition 1.1 ([3]) *A* **central region** *is the largest connected region covered with cycles surrounding O.*

Definition 1.2 ([3]) *A* **period annulus** *is every connected region covered with nontrivial concentric cycles.*

Definition 1.3 *We will call a period annulus associated with a central region* **a trivial period annulus**. *Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center.*

Definition 1.4 *Respectively a period annulus enclosing several (more than one) critical points will be called a* **nontrivial period annulus**.

2 Nontrivial period annuli

Consider equation (1), where the function $g(x)$ is polynomial

$$g(x) = -x(x+3)(x+2.2)(x+1.9)(x+0.8)(x-0.3)(x-1.5)(x-2.3)(x-2.9), \quad (5)$$

like in Fig. 2.1.

Proposition 2.1 *Critical points of the system (4) are "saddles" and "centers" which alternate.*

The primitive $G(x) = \int_0^x g(s)\,ds$ may have multiple maxima (see Fig. 2.1). It is easy to observe that the equivalent differential system (4) has centers at the points $(m_i, 0)$ and saddle points at $(M_j, 0)$, where m_i and M_j are points of local minima and maxima respectively.

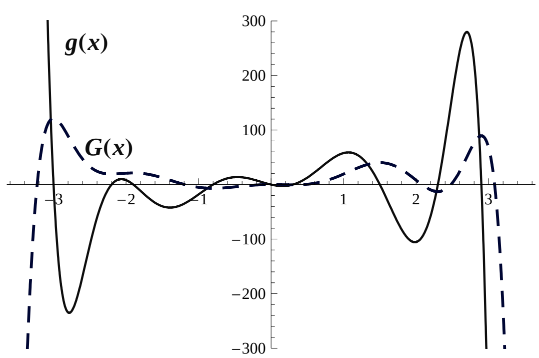
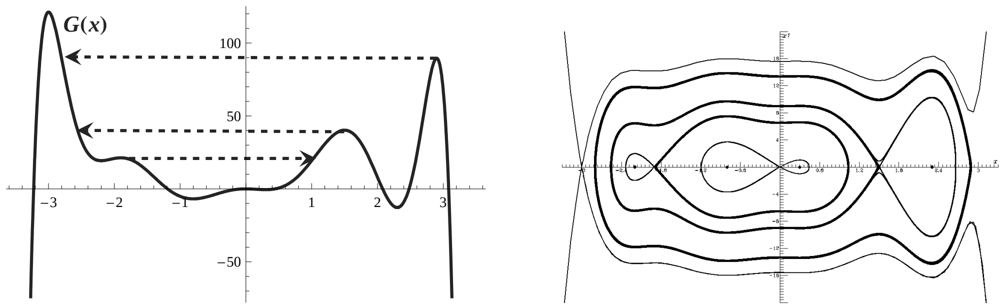


Fig. 2.1 Function $g(x)$ and its primitive function $G(x)$.

Theorem 2.1 *([4]) Let M_1 and M_2 ($M_1 < M_2$) be non-neighbouring points of maximum of the function $G(x)$. Suppose that any other local maximum of $G(x)$ in the interval (M_1, M_2) is (strictly) less than $min\{G(M_1); G(M_2)\}$.*

Then there exists at least one nontrivial period annulus.

Figure 2.2 a visualizes Theorem 2.1 and as the result there are 3 nontrivial period annuli (their x-locations are shown by arrows). The typical phase portrait for system (4) is given in Fig. 2.2 b. Also in this figure three nontrivial period annuli are depicted which enclose several (respectively 3, 5 or 7) critical points.



a. b.

Fig. 2.2 **a.** Primitive function; **b.** phase portrait.

3 Neumann problem

Consider equation (1) together with conditions (3).

Definition 3.1 *Intersection of a period annulus* $\mathbb{P}$ *with the positive half-plane* (x, x'), $x > 0$, *will be called a positive part of* $\mathbb{P}$ *and denoted* $\mathbb{P}_+$.

Definition 3.2 *Intersection of a period annulus* $\mathbb{P}$ *with the negative half-plane* (x, x'), $x < 0$, *will be called a negative part of* $\mathbb{P}$ *and denoted* $\mathbb{P}_-$.

Let us introduce the notation.

If $\mathbb{P}_+$ is a positive part and if $\gamma = x'(0) > 0$ varies between min and max values, then $T_+(\gamma)$ is a time needed for $(x(t), x'(t))$ to move from a position $(0, \gamma)$ to $(0, -\gamma)$.

If $\mathbb{P}_-$ is a negative part and if $-\gamma = x'(0) < 0$ varies between min and max values, then $T_-(-\gamma)$ is a time needed for $(x(t), x'(t))$ to move from a position $(0, -\gamma)$ to $(0, \gamma)$.

3.1 Trivial period annuli

Theorem 3.1 [2] *Let the conditions*

$$k^2\pi^2 < |g_x(m_i)| < (k+1)^2\pi^2 \tag{6}$$

hold. Then the Neumann BVP (1), (3) has at least $2k$ *nonconstant periodic solutions.*

Function $g(x)$ has 4 minima, so there are 4 trivial period annuli (see Fig. 3.1).

By computing of $g_x(-2.2) = 156.923$, $g_x(-0.8) = 78.6534$, $g_x(0.3) = 37.3745$, $g_x(2.3) = 685.644$ we get that the conditions

$$\begin{array}{rcrcl} 3^2\pi^2 & < & 156.9230 & < & 4^2\pi^2, \\ 2^2\pi^2 & < & 78.6534 & < & 3^2\pi^2, \\ 1^2\pi^2 & < & 37.3745 & < & 2^2\pi^2, \\ 8^2\pi^2 & < & 685.6440 & < & 9^2\pi^2 \end{array}$$

hold. Then by Theorem 3.1 the Neumann BVP (1), (3) has at least 28 solutions (see Fig. 3.2)

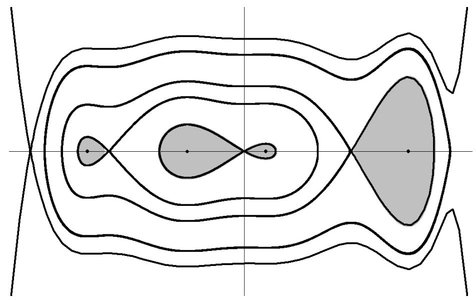


Fig. 3.1 Phase portrait with trivial period annuli.

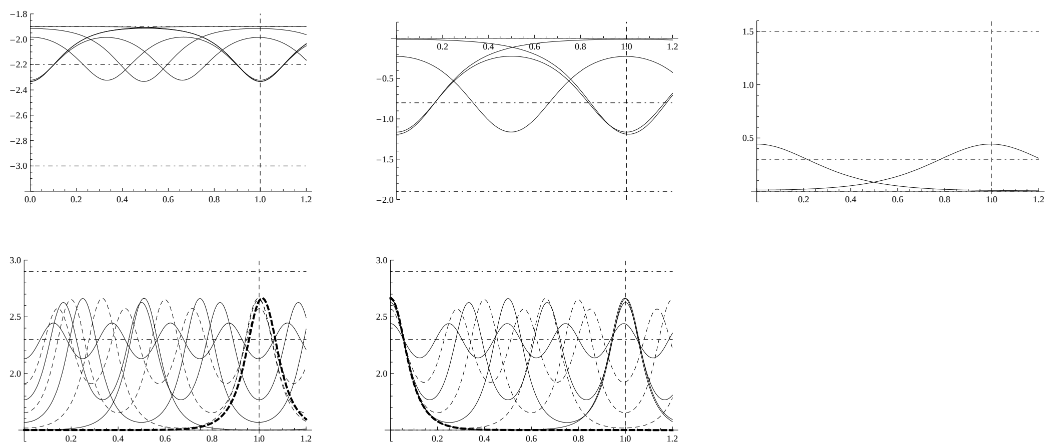


Fig. 3.2 Solutions of the BVP (1), (3).

3.2 Nontrivial period annuli

Theorem 3.2 *Suppose equation (1) has multiple period annuli* $\mathbb{P}_i$, $i = 1, 2, \ldots, m$. *Let* $\mathbb{P}_x$ *be intersection of a period annulus* $\mathbb{P}_i$ *with the* x*-axis (this intersection is a sum of two open intervals* I_1 *and* I_2*). Let* $T_i(x)$ *denote the time needed for a point* $(x, 0)$, $x \in I_1$, *to move along a trajectory of the equation to its position* $(x_*, 0)$ *in* I_2, $x'(t)$ *being nonnegative. Denote* $T_{i\min} = \min\{T_i(x) : \, x \in I_1\}$.

Suppose that positive integers k_i *satisfy the relations*

$$k_i T_{i\min} < 1 < (k_i + 1) T_{i\min}. \tag{7}$$

Then the problem (1), (3) has at least $4(k_1 + \ldots + k_m)$ *solutions.*

Remark 3.1. The half-period $T(x)$ is given by the formula

$$T(x) = \frac{1}{\sqrt{2}} \int_x^{x_1} \frac{ds}{\sqrt{G(s) - G(x)}},$$

where x_1 is the first zero of the function $\theta(s) = G(s) - G(x) = 0$ to the right of x.

The first period annulus encloses 7 critical points.

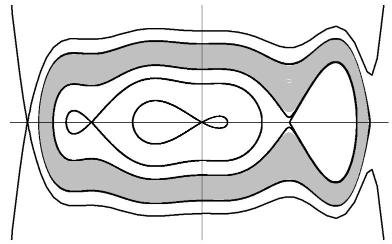
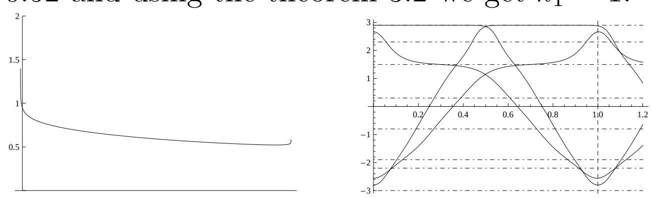


Fig. 3.3 The first period annulus (grey color) in the phase plane.

Calculating $T_{1min} = 0.52$ and using the theorem 3.2 we get $k_1 = 1$.



a. b.

Fig. 3.4 **a.** Graphs of $T(x)$ ($x(0) \in (2.663; 2.9)$); **b.** solutions of the BVP (1), (3).

The second period annulus encloses 5 critical points.

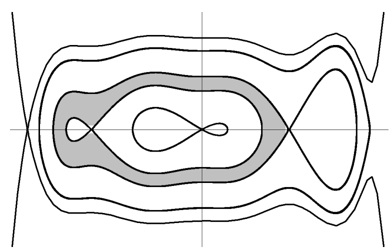
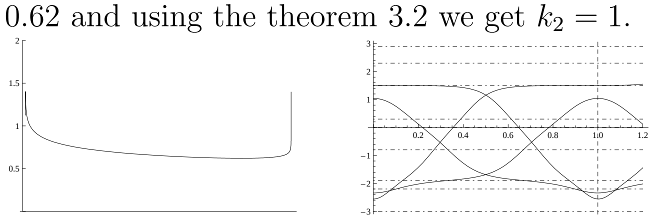


Fig. 3.5 The second period annulus (grey color).

Calculating $T_{2min} = 0.62$ and using the theorem 3.2 we get $k_2 = 1$.



a. b.

Fig. 3.6 **a.** Graphs of $T(x)$ ($x(0) \in (1.035; 1.5)$); **b.** solutions of the BVP (1), (3).

The third period annulus encloses 3 critical points.

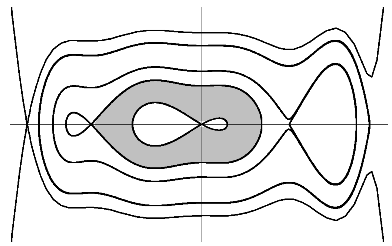
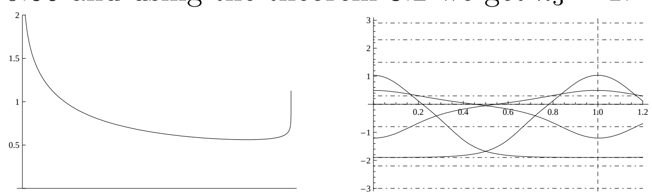


Fig. 3.7 The third period annulus (grey color).

Calculating $T_{3min} = 0.56$ and using the theorem 3.2 we get $k_3 = 1$.



a. b.

Fig. 3.8 **a.** Graphs of $T(x)$ $(x(0) \in (0.443; 1.035))$; **b.** solutions of the BVP (1), (3).

By Theorem 3.2 the Neumann BVP has $4(1+1+1) = 12$ solutions (nontrivial period annuli). So the Neumann BVP has 40 solutions (counting all solutions in the trivial and nontrivial period annuli).

4 Dirichlet problem

Consider Dirichlet problem (1), (2).

The time that is needed to go from a point $(0, \gamma)$ to the point $(0, -\gamma)$ by trajectory is given by the formula

$$T = 2\int_0^{x_1} \frac{dx}{\sqrt{\gamma^2 - 2G(x)}},$$

where $\gamma = \sqrt{2G(x_1)}$, x_1 is a point of intersection of the graph of a solution with the x-axis.

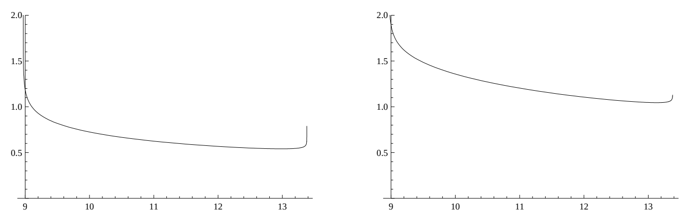
Suppose that a period annulus $\mathbb{P}$ has both negative and positive parts $\mathbb{P}_-$ and $\mathbb{P}_+$. Denote $T_+(\gamma)$ the time needed to go from a point $(0, \gamma)$ to the symmetrical point $(0, -\gamma)$ along the trajectory, and $T_-(-\gamma)$ the time to go from a point $(0, -\gamma)$ to $(0, \gamma)$, $\gamma > 0$. If γ tends to $\gamma_{homoclinic}$ and the positive part $(x > 0)$ of a homoclinic solution contains a saddle point then $T_+(\gamma) \to +\infty$. The function $T_-(-\gamma)$ tends to some finite value $T_-(-\gamma_{homoclinic})$, because the negative part $(x < 0)$ of a homoclinic solution does not contain critical points. If a saddle point of homoclinic solution is in the negative part then $T_-(-\gamma)$ tends to $+\infty$ as $-\gamma$ tends to $-\gamma_{homoclinic}$ and $T_+(\gamma)$ tends to a finite value $T_+(\gamma_{homoclinic})$.

4.1 Nontrivial period annuli

The first period annulus (see Fig. 3.3)

Intersection of the first period annulus with the x'-axis consists of two intervals $(\gamma_{2homoclinic}, \gamma_{1homoclinic})$ and the symmetrical one $(-\gamma_{1homoclinic}, -\gamma_{2homoclinic})$. Homoclinic solutions with the initial values at $(0, \gamma_{2homoclinic})$ and $(0, \gamma_{1homoclinic})$ form inner and outer boundaries of the first period annulus. Both have saddle points in the positive part $\mathbb{P}_+$. That is why $T(\gamma)$ tends to infinity as γ tends to $\gamma_{2homoclinic}$ or $\gamma_{1homoclinic}$.

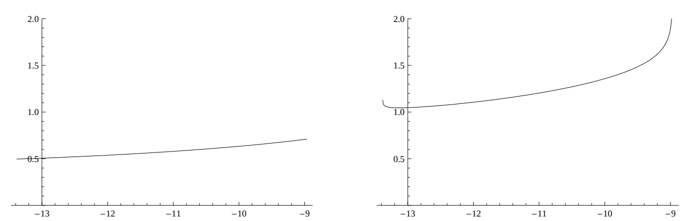
One has that $T_-(\gamma_{1homoclinic})$ and $T_-(\gamma_{2homoclinic})$ are finite, moreover $T_-(\gamma_{1homoclinic}) < T_-(\gamma_{2homoclinic})$. We consider the time $T(\gamma)$ which is needed to go from a point $(0, \gamma)$, $\gamma_{2homoclinic} < \gamma < \gamma_{1homoclinic}$, to itself along the trajectory. By calculating we get the minimal time $T_+ + T_- = 1.1224 > 1$. So there are not Dirichlet solutions with $T_+ + T_- = 1$. On the other hand, $T_+ min = 0.436609$. In this case there are 2 solutions.



a. b.

Fig. 4.1 **a.** Time T_+; **b.** time $T_+ + T_-$.

Similarly for moving from point $(0; -\gamma)$ we get: $T_- + T_+ = 1.1224 > 1$ (therefore no solutions with $T_- + T_+ = 1$) and $T_- max = T_-(\gamma_{2homoclinic}) = 0.496375$. So there are no solutions of the Dirichlet problem with $x'(0) \in (\gamma_{2homoclinic}, \gamma_{1homoclinic})$.



a. b.

Fig. 4.2 **a.** Time T_-; **b.** time $T_- + T_+$.

Both cases are shown in Fig. 4.3.

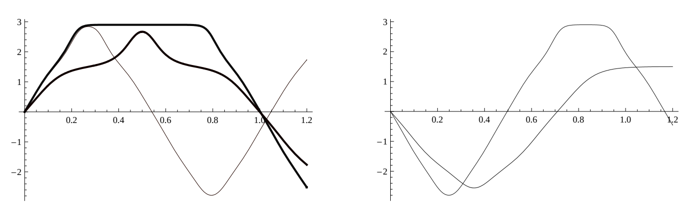
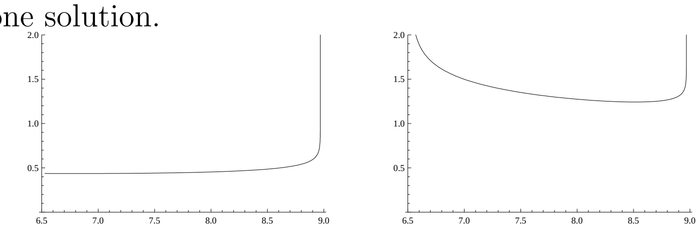


Fig. 4.3 Solutions for problem (1), (2) (bold lines).

The second period annulus (see Fig. 3.5)

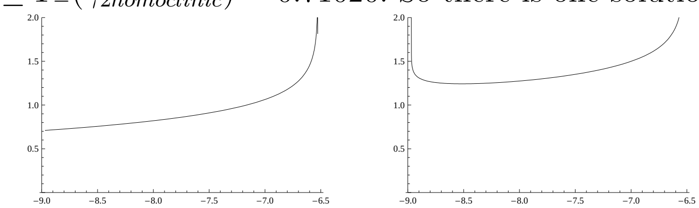
The second period annulus has two homoclinic solutions (the second one encloses the third) with the property: $T_{-}(\gamma_{2homoclinic})$ and $T_{+}(\gamma_{3homoclinic})$ are finite. We consider the time that is needed to go from a point $(0;\gamma)$ to the same point by trajectory where $\gamma_{3homoclinic} < \gamma < \gamma_{2homoclinic}$. By calculating we get the minimal time $T_{+}+T_{-} = 1.42962 > 1$. So there are no Dirichlet solutions with $T_{+}+T_{-} = 1$. Then consider the time that is needed to go from a point $(0;\gamma)$ to the point $(0;-\gamma)$, $T_{+min} \leq T_{+}(\gamma_{3homoclinic}) = 0.436609$. In this case there is one solution.



a. b.

Fig. 4.4 **a.** Time T_{+}; **b.** time $T_{+}+T_{-}$.

Similarly for moving from a point $(0;-\gamma)$ we get: $T_{-}+T_{+} = 1.42962 > 1$ (whitout solutions) and $T_{-min} \leq T_{-}(\gamma_{2homoclinic}) = 0.71020$. So there is one solution.



a. b.

Fig. 4.5 **a.** Time T_{-}; **b.** time $T_{-}+T_{+}$.

Both cases are shown in Fig. 4.6.

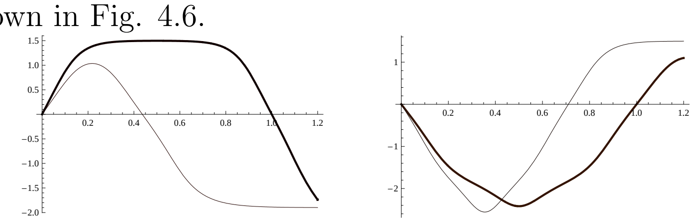
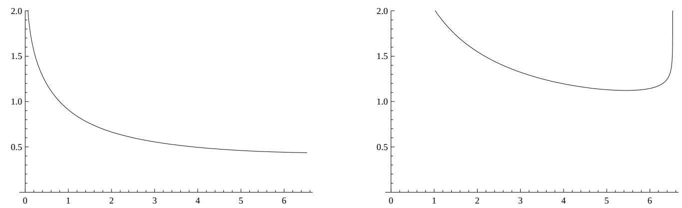


Fig. 4.6 Solutions for the problem (1), (2) (bold lines).

The third period annulus (see Fig. 3.7)

The third period annulus is bounded by outer homoclinic solution ($\gamma = \gamma_{3homoclinic}$). The inner boundary is figure-eight which is formed by two homoclinic solutions. They share common saddle point at $(0,0)$. One has that $T_{+}(\gamma_{3homoclinic})$ is finite. We consider the time that is needed to go from a point $(0;\gamma)$ to the same point by trajectory where $0 = \gamma_{4homoclinic} < \gamma < \gamma_{3homoclinic}$. By calculating we get the minimal time $T_{+}+T_{-} = 1.1224 > 1$. So there are not Dirichlet solutions with $T_{+}+T_{-} = 1$. Then consider the time that is needed to go from a point $(0;\gamma)$ to the point $(0;-\gamma)$, $T_{+min} \leq T_{+}(\gamma_{3homoclinic}) = 0.436609$. In this case there is one solution.



a. b.

Fig. 4.7 **a.** Time T_+; **b.** time $T_+ + T_-$.

Similarly for moving from a point $(0; -\gamma)$ we get: $T_- + T_+ = 1.1224 > 1$ (no solutions with $T_- + T_+ = 1$) and $T_{-min} = 0.669609$. So there are 2 solutions.

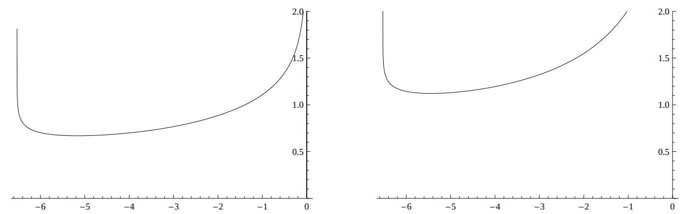


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Fig. 4.8 **a.** Time T_-; **b.** time $T_- + T_+$.

Both cases are shown in Fig. 4.9.

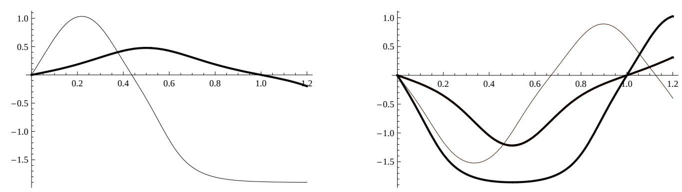

Fig. 4.9 Solutions for the problem (1), (2) (bold lines).

The Dirichlet problem (1), (2) has 7 solutions.

5 Comparison and conclusions

In this paper we considered specific the second order equation where $g(x)$ is a polynomial of ninth order. Its derivative is the tenth order polynomial with five local maxima. There are three pairs of nonneighboring maxima such that the inner maxima are less than outer ones. This ensures the existence of three period annuli which are continua of period solutions. The numeric analysis of periods shows that there exist 7 solutions which satisfy the Dirichlet boundary conditions $x(0) = 0$, $x(1) = 0$.

Solutions of the Neumann problem ($x'(0) = 0$, $x'(1) = 0$) are numerous. One of the reasons is that all critical points of the system $x' = y$, $y' = -g(x)$ are located on the x-axis. Any of these critical points is surrounded by closed trajectories which form regions containing multiple solutions of the Neumann problem. There are two types of solutions to the Neumann problem.

The first type solutions form orbits which are contained in the central regions (trivial period annuli) around the critical points of the type center (there are four). There are 28 solutions of the first kind.

The second type solutions have orbits which are contained in nontrivial period annuli (they are three).

The number of solutions of the Neumann problem in any period annulus is dependent on the minimal period of solutions in it.

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С. Атслега. Краевые задачи Дирихле и Неймана для одного уравнения.

Аннотация. Рассматривается специфическое автономное уравнение второго порядка, имеющее периодические кольца. Эти периодические решения содержат также решения задач Дирихле и Неймана, которые подсчитываются и дается теоретическое обоснование числа решений. Проведено сравнение двух краевых задач.

УДК 517.927

S. Atslēga. Dirihlē un Neimana problēmas vienam vienādojumam.

Anotācija. Apskatīts otrās kārtas specifisks autonoms vienādojums, kuram ir periodiskie gredzeni. Šie periodiskie gredzeni satur arī Dirihlē un Neimana problēmas atrisinājumus. Tiek aprēķināti un doti teorētiskie pamatojumi par atrisinājumu skaitu. Salidzinātas divas problēmas.

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