On Nonlinear Spectrum for some Nonlocal Boundary Value Problem

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Summary. We construct Fučik spectrum for specific differential equation. This spectrum differs essentially from the known ones.

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1 Introduction

Investigations oh Fučik spectra have started in sixties of XX century. Let us mention the work [5] and the bibliography therein. Of the recent works let us mention [7], [8]. The Fučik spectra have been investigated for the second order equation with different two-points boundary conditions. Less works are about the higher order problems.

Our goal is to get formulas for the second order BVP

$$x'' = -2\mu^4 x^{3+} + 2\lambda^4 x^{3-}, \quad \mu, \lambda \geq 0 \tag{1.1}$$

with the boundary conditions

$$x(0) = 0, \quad \int_0^1 x(s)ds = 0. \tag{1.2}$$

To the best of our knowledge Fučik spectra for problems with nonlocal boundary conditions were not considered previously.

In this paper we provide explicit formulas for Fučik spectrum of the problem (1.1), (1.2). This spectrum differs essentially from the classical one. These formulas were carried out using the functions, which are known as the lemniscatic functions

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[9]. So lemniscatic sine and cosine will be denoted $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$ respectively. The formulas for relations between lemniscatic functions and their derivatives are known [2].

This paper is organized as follows.

In Section 2 we present results on the Fučik spectrum for the problem (1.1) with boundary conditions $x(0) = x(1) = 0$. In Section 3 we consider the problem (1.1), (1.2) and construct the Fučik spectrum for this problem. This is the main result of the work. Connection between the spectra are discussed in Section 4.

2 The second order problem with the boundary conditions $\mathbf{x(0) = x(1) = 0}$

Consider the equation

$$x'' = -2\mu^4 x^{3+} + 2\lambda^4 x^{3-}, \quad \mu, \lambda \geq 0, \tag{2.1}$$

$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\},$$

with the boundary conditions

$$x(0) = x(1) = 0, \quad |x'(0)| = 1. \tag{2.2}$$

Definition 1 *The spectrum is a set of points* (λ, μ) *such that the problem* (2.1), (2.2) *has nontrivial solutions.*

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$ according to the number of zeroes of the derivative of a solution to the problem (2.1), (2.2) in the interval $(0, 1)$.

Proposition 1 *The Fučik spectrum consists of the set of curves*
$F_i^+ = \{(\lambda, \mu) |\, x'(0) = 1,$ *the nontrivial solution of the problem*$(2.1), (2.2)$ $x(t)$ *has exactly* i *zeroes in* $(0, 1)\}$;
$F_i^- = \{(\lambda, \mu) |\, x'(0) = -1,$ *the nontrivial solution of the problem*$(2.1), (2.2)$ $x(t)$ *has exactly* i *zeroes in* $(0, 1)\}$.

Theorem 2.1 ([8], subsection 3.2.1) *The Fučik spectrum for the problem* (2.1), (2.2) *consists of the branches given by*

$$F_0^+ = \Big\{(\lambda, 2A)\Big\},$$

$$F_0^- = \Big\{(2A, \mu)\Big\},$$

$$F_{2i-1}^+ = \Big\{(\lambda, \mu) \Big|\, i\frac{2A}{\mu} + i\frac{2A}{\lambda} = 1\Big\},$$

$$F_{2i}^+ = \Big\{(\lambda, \mu) \Big|\, (i+1)\frac{2A}{\mu} + i\frac{2A}{\lambda} = 1\Big\},$$

$$F_{2i-1}^{-} = \left\{(\lambda,\mu)\Big|\ i\frac{2A}{\mu} + i\frac{2A}{\lambda} = 1\right\},$$

$$F_{2i}^{-} = \left\{(\lambda,\mu)\Big|\ i\frac{2A}{\mu} + (i+1)\frac{2A}{\lambda} = 1\right\},$$

where $A = \int\limits_0^1 \frac{ds}{\sqrt{1-s^4}}$, $i = 1, 2, \ldots$.

Some first branches of the spectrum to the problem (2.1), (2.2) are depicted in Figure 1.

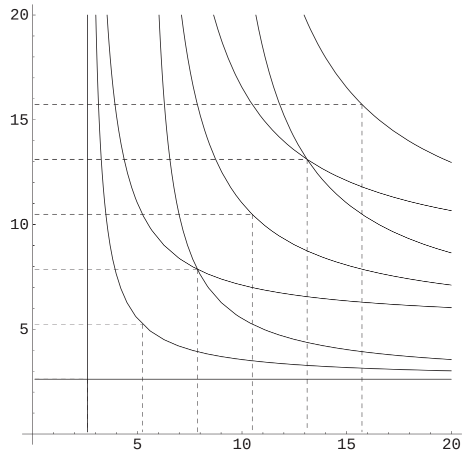

Figure 1: The spectrum for the problem (2.1), (2.2).

3 The second order problem with nonlocal boundary condition

Consider the equation

$$x'' = -2\mu^4 x^{3+} + 2\lambda^4 x^{3-}, \quad \mu, \lambda \geq 0, \tag{3.1}$$

$$x^{+} = \max\{x, 0\}, \quad x^{-} = \max\{-x, 0\},$$

with the boundary conditions

$$x(0) = 0, \quad \int\limits_0^1 x(s)ds = 0. \tag{3.2}$$

Decomposition of the spectrum for the problem (3.1), (3.2) into branches F_i^{+} and F_i^{-} ($i = 1, 2, \ldots$) is the same as that for the problem (2.1), (2.2).

The next theorem is the main result of this work.

Theorem 3.1 *The Fučik spectrum for the problem* (3.1), (3.2) *consists of the branches given by*

$$
\begin{aligned}
F^+_{2i-1} &= \Big\{(\lambda,\mu)\Big|\ \frac{2i\lambda}{\mu}\frac{\pi}{4} - \frac{(2i-1)\mu}{\lambda}\frac{\pi}{4} - \frac{\mu \arctan \mathrm{cl}\,(\lambda - \lambda\frac{2A}{\mu}i + 2Ai)}{\lambda} = 0,\\
&\quad i\frac{2A}{\mu} + (i-1)\frac{2A}{\lambda} \le 1,\ i\frac{2A}{\mu} + i\frac{2A}{\lambda} > 1\Big\},\\
F^+_{2i} &= \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\lambda}{\mu}\frac{\pi}{4} - \frac{2i\mu}{\lambda}\frac{\pi}{4} - \frac{\lambda \arctan \mathrm{cl}\,(\mu - \mu\frac{2A}{\lambda}i + 2Ai)}{\mu} = 0,\\
&\quad i\frac{2A}{\mu} + i\frac{2A}{\lambda} \le 1,\ (i+1)\frac{2A}{\mu} + i\frac{2A}{\lambda} > 1\Big\},\\
F^-_{2i-1} &= \Big\{(\lambda,\mu)\Big|\ \frac{2i\mu}{\lambda}\frac{\pi}{4} - \frac{(2i-1)\lambda}{\mu}\frac{\pi}{4} - \frac{\lambda \arctan \mathrm{cl}\,(\mu - \mu\frac{2A}{\lambda}i + 2Ai)}{\mu} = 0,\\
&\quad (i-1)\frac{2a}{\mu} + i\frac{2A}{\lambda} \le 1,\ i\frac{2A}{\mu} + i\frac{2A}{\lambda} < 1\Big\},\\
F^-_{2i} &= \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\mu}{\lambda}\frac{\pi}{4} - \frac{2i\lambda}{\mu}\frac{\pi}{4} - \frac{\mu \arctan \mathrm{cl}\,(\lambda - \lambda\frac{2A}{\mu}i + 2Ai)}{\lambda} = 0,\\
&\quad i\frac{2A}{\mu} + i\frac{2A}{\lambda} \le 1,\ i\frac{2A}{\mu} + (i+1)\frac{2A}{\lambda}\ 1\Big\},
\end{aligned}
$$

where $\mathrm{cl}\,(t)$ *is the lemniscatic cosine function,* $A = \int\limits_0^1 \frac{ds}{\sqrt{1-s^4}}$, $i = 1, 2, \ldots$.

Proof. Consider the problem (3.1), (3.2).

It is clear that $x(t)$ must have zeroes in $(0,1)$. That is why $F_0^{\pm} = \emptyset$.

We will prove the theorem for the case of F_1^+. Suppose that $(\lambda,\mu) \in F_1^+$ and let $x(t)$ be a respective nontrivial solution of the problem (3.1), (3.2). The solution has only one zero in $(0,1)$ and $x'(0) = 1$. Let this zero be denoted by τ.

Consider a solution of the problem (3.1), (3.2) in the interval $(0,\tau)$ and in the interval $(\tau,1)$. We obtain that the equation (3.1) in these intervals reduces to the Emden - Fowler differential equations. So in the interval $(0,\tau)$ we have the problem $x'' = -2\mu^4x^3$ with boundary conditions $x(0) = x(\tau) = 0$, but in the interval $(\tau,1)$ we have the problem $x'' = -2\lambda^4x^3$ with boundary condition $x(\tau) = 0$. In view of (3.2) a solution $x(t)$ must satisfy the condition

$$\int\limits_0^\tau x(s)ds = \Big|\int\limits_\tau^1 x(s)ds\Big|. \tag{3.3}$$

Since $x(t) = \frac{1}{\mu}$, $x(\tau) = 0$ and the first positive zero of $\mathrm{sl}\,(\xi)$ is at $2A$, we obtain $\tau = \frac{2A}{\mu}$. In view of this equality it is easy to get that $\int\limits_0^\tau x(s)ds = \frac{1}{\mu^2}\big(\frac{\pi}{4} - \arctan \mathrm{cl}\, 2A\big) = \frac{2}{\mu^2}\frac{\pi}{4}$. We use here the formula (4.4) from [1, p.27]. We have also

$$x'\Big(\frac{2A}{\mu}\Big) = -1. \tag{3.4}$$

Now we consider a solution of the problem (3.1), (3.2) in $[\tau, 1]$.Since $x(t) = -\frac{1}{\lambda}\mathrm{sl}\,(\lambda t - \lambda\frac{2A}{\mu}$ we obtain $\left|\int\limits_{\tau}^{1} x(s)ds\right| = \frac{1}{\lambda^2}\big(\frac{\pi}{4} - \arctan \mathrm{cl}\,(\lambda - \lambda\frac{2A}{\mu})\big)$.

We have also that

$$x'\Big(\frac{2A}{\mu}\Big) = -1. \qquad (3.5)$$

In view of the last equality and (3.3) we obtain $\frac{2}{\mu^2}\frac{\pi}{4} = \frac{1}{\lambda^2}\big(\frac{\pi}{4} - \arctan \mathrm{cl}\,(\lambda - \lambda\frac{2A}{\mu})\big)$. Multiplying by $\mu\lambda$, we obtain

$$\frac{2\lambda}{\mu}\frac{\pi}{4} - \frac{\mu}{\lambda}\frac{\pi}{4} + \frac{\mu \arctan \mathrm{cl}\,(\lambda - \lambda\frac{2A}{\mu})}{\lambda} = 0. \qquad (3.6)$$

Considering the solution of the problem (3.1), (3.2) it is easy to prove that $0 < \frac{2A}{\mu} < 1 < \frac{2A}{\mu} + \frac{2A}{\lambda}$.

This result and (3.6) prove the theorem for the case of F_1^+. The proof for other branches is analogous. □

Visualization of the spectrum to the problem (3.1), (3.2) is given in Figure 2.

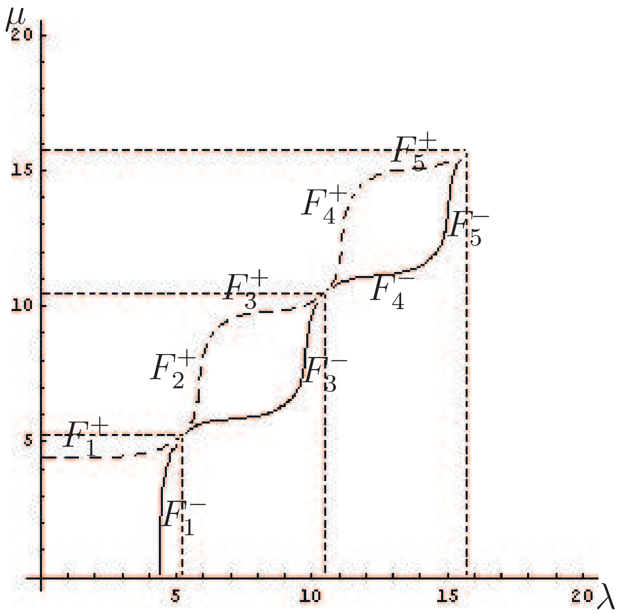

Figure 2: The spectrum for the problem (2.1), (2.2).

4 Comparison

Now we consider the equation (3.1) with boundary conditions

$$x(0) = 0, \;\; (1-\alpha)x(t) + \alpha\int\limits_0^1 x(s)ds = 0, \;\; \alpha \in [0; 1]. \qquad (4.1)$$

Theorem 4.1 *The Fučik spectrum* $\sum_\alpha = \bigcup_{i=0}^{+\infty} F_i^{\pm}$ *for the problem* (3.1), (4.1), *where meaning of the notation is the same as earlier, consists of the branches given by (where* $i = 1, 2, \ldots$*)*

$$
\begin{aligned}
F_{2i-1}^{+} &= \Big\{(\lambda,\mu)\Big| \; \frac{2i\lambda}{\mu}\frac{\pi}{4}\alpha - \frac{(2i-1)\mu}{\lambda}\frac{\pi}{4}\alpha - \frac{\mu\alpha \arctan \operatorname{cl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right)}{\lambda} + \\
&+\mu \operatorname{sl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right) - \alpha\mu \operatorname{sl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right) = 0, \\
&i\frac{2A}{\mu} + (i-1)\frac{2A}{\lambda} \le 1,\; i\frac{2A}{\mu} + i\frac{2A}{\lambda} > 1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{2i}^{+} &= \Big\{(\lambda,\mu)\Big| \; \frac{(2i+1)\lambda}{\mu}\frac{\pi}{4}\alpha - \frac{2i\mu}{\lambda}\frac{\pi}{4}\alpha - -\frac{\lambda\alpha \arctan \operatorname{cl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right)}{\mu} + \\
&+\lambda \operatorname{sl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right) - \alpha\lambda \operatorname{sl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right) = 0, \\
&i\frac{2A}{\mu} + i\frac{2A}{\lambda} \le 1,\; (i+1)\frac{2A}{\mu} + i\frac{2A}{\lambda} > 1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{2i-1}^{-} &= \Big\{(\lambda,\mu)\Big| \; \frac{2i\mu}{\lambda}\frac{\pi}{4}\alpha - \frac{(2i-1)\lambda}{\mu}\frac{\pi}{4}\alpha - \frac{\lambda\alpha \arctan \operatorname{cl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right)}{\mu} + \\
&+\lambda \operatorname{sl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right) - \alpha\lambda \operatorname{sl}\left(\mu - \mu\frac{2Ai}{\lambda} + 2Ai\right) = 0, \\
&(i-1)\frac{2A}{\mu} + i\frac{2A}{\lambda} \le 1,\; i\frac{2A}{\mu} + i\frac{2A}{\lambda} > 1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{2i}^{-} &= \Big\{(\lambda,\mu)\Big| \; \frac{(2i+1)\mu}{\lambda}\frac{\pi}{4}\alpha - \frac{2i\lambda}{\mu}\frac{\pi}{4}\alpha - \frac{\mu\alpha \arctan \operatorname{cl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right)}{\lambda} + \\
&+\mu \operatorname{sl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right) - \alpha\mu \operatorname{sl}\left(\lambda - \lambda\frac{2Ai}{\mu} + 2Ai\right) = 0, \\
&i\frac{2A}{\mu} + i\frac{2A}{\lambda} \le 1,\; i\frac{2A}{\mu} + (i+1)\frac{2A}{\lambda} > 1\Big\}.
\end{aligned}
$$

Proof. The proof of Theorem is analogous to that of Theorem 3.1. □

Remark 4.1 *If* $\alpha = 0$ *we obtain the problem* (2.1), (2.2). *In case of* $\alpha = 1$ *we have the problem* (3.1), (3.2).

The branches $F_1^{\pm}$ to $F_5^{\pm}$ of the spectrum for the problem (3.1), (4.1) for several values of α are depicted in Figures 3 and 4 in the case of $a = 0$, $b = 1$.

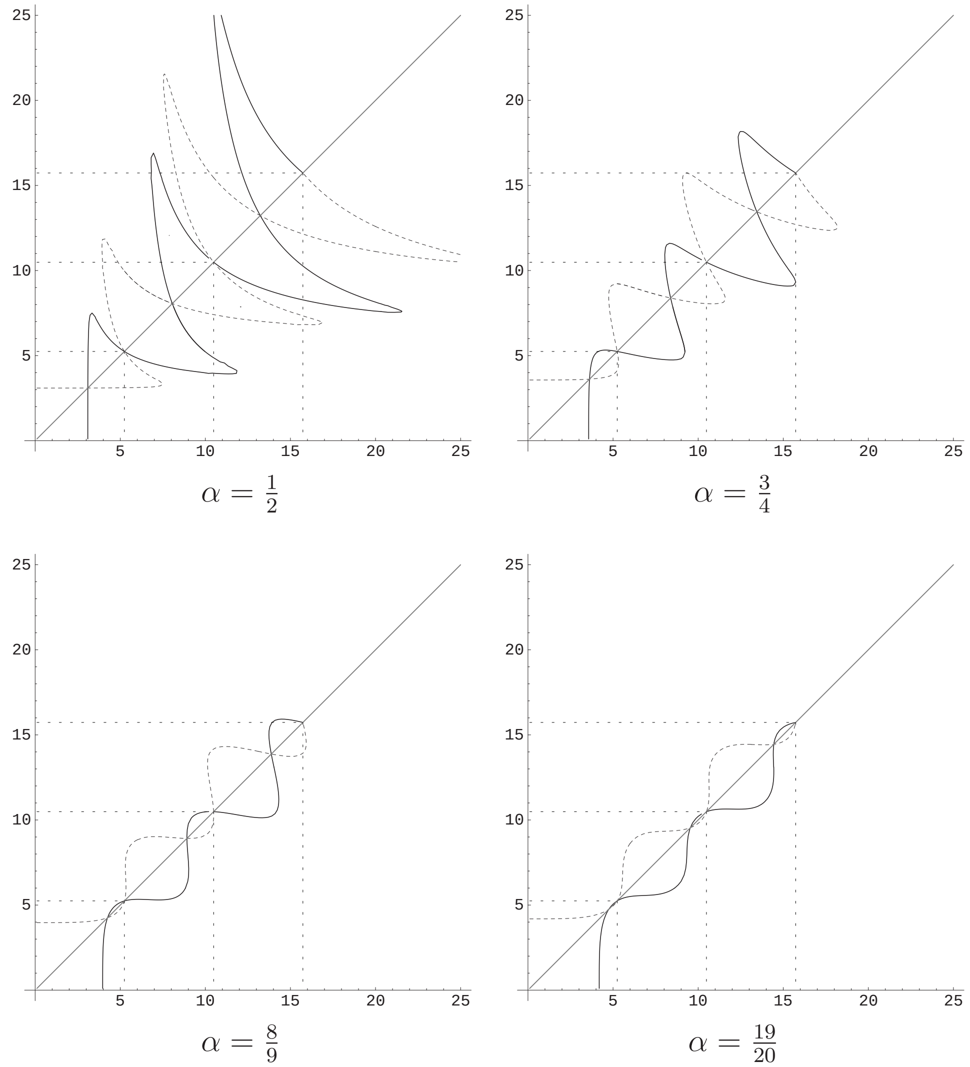

Figure 3: The Fučik spectrum for the problem (3.1), (4.1) for same α values

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Н. Сергеева. Спектры Фучика для одной краевой задачи с нелокальным граничным условием.

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Anotācija. Mēs konstruējam Fučika spektrus kādam nelineāram otrās kārtas diferenciālvienādojumam. Mūsu spektri atšķirās no zināmiem.

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