Multiple solutions for Φ-Laplacian equations with the Dirichlet boundary conditions

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Summary. We consider Φ-Laplacian type equation $\frac{d}{dt}\Phi(x') + f(t,x) = 0$ together with the Dirichlet boundary conditions. This equation is reduced to two-dimensional differential system. The quasi-linearization process is applied to obtain the conditions for existence of multiple solutions.

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1 Introduction

In this article we consider the Φ-Laplacian type equation

$$\frac{d}{dt}\Phi(x') + f(t,x) = 0 \tag{1}$$

together with the boundary conditions

$$x(0) = 0, \quad x(1) = 0. \tag{2}$$

This equation (even in a greater generality) was intensively studied in the last time literature ([1], [2] and references therein). If $\Phi(x') = x'$ then it reduces to equation

$$x'' + f(t,x) = 0. \tag{3}$$

The equation (1) is also the Euler equation for the functional

$$J(x) = \int_0^1 (\Psi(x') - F(t,x))\, dt, \tag{4}$$

where $\Phi(x') = \Psi'(x')$ and $f(t,x) = F_x(t,x)$.

Our aim in this paper is to obtain the multiplicity results using the quasi-linearization process described in [10], [9], [11]. For this we rewrite equation (1) as a two-dimensional differential system of the form (5) and apply the quasi-linearization process.

In section 2 we give definitions. In section 3 the main result is proved concerning solvability of a quasi-linear boundary value problem. Section 4 contains application of the main result and of the quasi-linearization process to a nonlinear system.

Let us describe the quasi-linearization process briefly. Consider a system of the form

$$\begin{cases} x' = f_1(y), \\ y' = f_2(t,x). \end{cases} \tag{5}$$

We classify first linear differential systems of the form

$$\begin{cases} x' - k^2 y = 0, \\ y' + l^2 x = 0 \end{cases} \tag{6}$$

with respect to the boundary conditions (2). Then we introduce a notion of the type of a solution to nonlinear BVP. It is shown that quasi-linear BVP

$$\begin{cases} x' - k^2 y = F_1(y), \\ y' + l^2 x = F_2(t,x), \end{cases} \tag{7}$$

where boundary conditions are of the form (2), has a solution (ξ,η) such that the type of (ξ,η) corresponds to the class of a linear part in (6). Then we discuss possible reduction of the problem (5), (2) to a quasi-linear one with certain linear part. If similar reduction is possible with respect to another essentially different linear part, then the problem (5), (2) has multiple solutions.

In the final section examples, calculations and illustrations are given which help to understand the approach.

2 Definitions

Consider the quasi-linear system (7). In order to classify the linear parts of (7) consider the homogeneous system (6) together with the boundary conditions (2).

Introduce polar coordinates as

$$x(t) = r(t)\,\sin\phi(t), \quad y(t) = r(t)\,\cos\phi(t). \tag{8}$$

Then the angular function $\phi(t)$ for (6) satisfies

$$\phi'(t) = l^2 \sin^2\phi(t) + k^2\cos^2\phi(t). \tag{9}$$

Notice that the quadratic form in (9) is positive definite and therefore the angular function $\phi(t)$ is monotonically increasing.

The boundary conditions (2) in polar coordinates take the form

$$\phi(0) = 0, \quad \phi(1) = \pi\, n, \quad n \in \mathbb{N}. \tag{10}$$

Definition 2.1 *A linear part in (6) is called i-nonresonant with respect to the boundary conditions (2) if the angular function $\phi(t)$, defined by the initial condition $\phi(0) = 0$, satisfies the inequalities*

$$i\,\pi < \phi(1) < (i+1)\,\pi, \quad i \in \{0,1,\ldots\}. \tag{11}$$

Remark 2.1. Evidently a linear part in the system (6) is i-nonresonant if $k^2l^2 \in \big(i^2\pi^2, (i+1)^2\pi^2\big)$. In other words, $\phi(t)$ takes exactly i values of the form $\pi\, n,\ t \in (0, 1]$.

Let (ξ, η) be a solution to the quasi-linear problem (7), (2). Consider also neighboring solutions (x, y) of the system (7) with the initial conditions such that $x(0) = 0$, $y(0) > \eta(0)$. In order to classify solutions of the BVP under considerations introduce (local) polar coordinates as

$$x(t) - \xi(t) = \rho(t)\, \sin\Theta(t; \rho_0), \quad y(t) - \eta(t) = \rho(t)\, \cos\Theta(t; \rho_0). \tag{12}$$

Definition 2.2 *Suppose that* $\rho_0 = \rho(0) > 0$ *and* $\Theta(0; \rho_0) = 0$. *We say that* (ξ, η) *is an* i*-type solution of the problem (7), (2) if for some small number* $\varepsilon > 0$ *the function* Θ *satisfies*

$$i\,\pi < \Theta(1; \rho_0) < (i + 1)\,\pi, \quad i \in \{0, 1, \ldots\} \tag{13}$$

for $\rho_0 \in (0, \varepsilon)$.

3 Main result for quasi-linear systems

This is the main part of the work. In the first subsection we prove main theorem on existence of an i-type solution to the quasi-linear problem with similar properties of a linear part. In the second subsection we provide consequences for the problem (1), (2). In the third subsection we recall the results by Jackson - Schrader - Knobloch - Erbe about the specific solutions to the boundary value problem

$$x'' + f(t, x) = 0, \quad x(0) = 0,\ x(1) = 0, \tag{14}$$

which in terminology of this article are 0-type solutions. We extend the above results to problems which have i-type solutions.

3.1 Quasi-linear problem

Consider quasi-linear system (7), where functions F_1, F_2 are continuous, bounded and satisfy the Lipschitz conditions with respect to y and x respectively. The Cauchy problems (7), $x(t_0) = A$, $y(t_0) = B$ are then uniquely solvable and solutions $(x(t), y(t))$ continuously depend on the initial data and on the right sides of the system (7).

The following result is true.

Lemma 3.1 *A set* S *of solutions to the problem (7), (2) is non-empty and compact in* $C^1_2([0, 1], R)$.

The proof is standard, using the Green'f function approach and the Arzela - Ascoli criterium ([4]).

Corollary 3.1 *A set of initial values* $(0, y(0))$ *of solutions to the problem (7), (2) is compact in* R.

Before to prove the main theorem, let us state the lemma.

Lemma 3.2 *Let (ξ, η) be a solution to the problem (7), (2). The angular function $\Theta(t; \rho_0)$ (defined in (12)) tends to $\varphi(t)$ (defined in 8) as ρ_0 tends to $+\infty$ uniformly in $t \in [0, 1]$.*

Proof. The normalized functions

$$u = \frac{1}{\rho_0}(x(t; \rho_0) - \xi(t)), \quad v\frac{1}{\rho_0}(y(t; \rho_0) - \eta(t))$$

satisfy the system

$$\begin{cases} u' - k^2 u = \dfrac{1}{\rho_0}[F_1(y) - F_1(\eta)], \\ \\ v' + l^2 v = \dfrac{1}{\rho_0}[F_2(t, x) - F_2(t, \xi)]. \end{cases} \tag{15}$$

The right sides in (15) tend to zero uniformly in $t \in [0, 1]$ as $\rho_0 \to +\infty$. The functions $(u(t), v(t))$ tend to a solution $(x_1(t), y_1(t))$ of the homogeneous system (6), which satisfies the initial conditions $\phi(0) = 0$, $r(0) = 1$ (equivalently $x_1(0) = 0$, $y_1(0) = 1$), where $\phi(t)$ is the angular function for $(x_1(t), y_1(t))$. Therefore $\Theta(t; \rho_0) \to \phi(t)$ as $\rho_0 \to +\infty$, uniformly in $t \in [0, 1]$. As a consequence, $\Theta(1; \rho_0)$ satisfies for given i the inequalities (13) together with $\phi(1)$. $\square$

The main theorem follows.

Theorem 3.1 *Suppose that the linear part in (7) is i-nonresonant with respect to the boundary conditions (2).*
Then the problem (7), (2) has an i-type solution.

Proof. Let (ξ, η) be a solution of the problem (7), (2) which has a maximal value of $\eta(0)$ among all solutions of the problem. Such a solution exists since a set $\{(0, \eta(0)) : (\xi, \eta) \in S\}$ is compact in R.

Consider the difference $(x(t)-\xi(t), y(t)-\eta(t))$, where (x, y) is a solution of the system (7) such that $x(0) = 0$ and $y(0) > \eta(0)$. We claim that (ξ, η) is an i-type solution to the problem. Suppose this is not true.

Consider the case $\Theta(1; \rho_0) = \pi(i+1)$ for $\rho_0 > 0$ and small enough. Then the respective (x, y) satisfy also the second boundary condition in (2) and therefore solve the BVP (7), (2). Then (ξ, η) is not a maximal solution in the above sense.

Suppose that $\Theta(1; \rho_0) > \pi(i+1)$ for small enough $\rho_0 > 0$. Then $\Theta(1; \rho_0)$ must become less than $\pi(i + 1)$ if ρ_0 goes to $+\infty$. Since all solutions of the system (7) extend to the interval $[0, 1]$ and are uniquely defined by the initial data, they continuously depend on the initial data and the same does the angular function $\Theta(1; \rho_0)$. Therefore there exists $\rho_0 > 0$ such that the respective (x, y) again satisfies also the second boundary condition in (2) and therefore solves the BVP (7), (2). Then (ξ, η) is not a maximal solution. The other cases can be treated similarly. $\square$

3.2 Φ-Laplacian equation

Consider the problem (1), (2). Let $\Phi(x') = y$ be $R \to R$ continuous function such that $x' = \Phi^{-1}(y)$ satisfies the Lipschitz condition. Let also $f(t, x)$ be continuous and

Lipschitzian with respect to x. Then the system

$$\begin{cases} x' = \ \ \Phi^{-1}(y), \\ y' = -f(t,x). \end{cases} \tag{16}$$

is equivalent to (1). Consider also modified system

$$\begin{cases} x' - k^2 y = \Phi^{-1} y - k^2 y =: F_1(y), \\ y' + l^2 x = l^2 x - f(t,x) =: F_2(t,x). \end{cases} \tag{17}$$

Let

$$\delta(x,y,z) = \begin{cases} z, & y > z, \\ y, & x \le y \le z, \\ x, & y < x. \end{cases}$$

Consider quasi-linear system

$$\begin{cases} x' - k^2 y = \hat{F}_1(y) := F_1(\delta(-N_1, y, N_1)), \\ y' + l^2 x = \hat{F}_2(t,x) =: F_2(t, \delta(-N_2, x, N_2)). \end{cases} \tag{18}$$

Due to properties of the functions Φ and f the right sides in (18) satisfy the Lipschitz condition and Theorem 3.1 is applicable.

Proposition 3.1 *The problem (18), (2) is solvable ([3]). Moreover, it has an i-type solution if the linear part in (18) is i-nonresonant with respect to the boundary conditions (2).*

The following assertion is evident.

Proposition 3.2 *If a solution $(x(t), y(t))$ to the problem (18), (2) satisfies the inequalities*

$$|y(t)| \le N_1, \quad |x(t)| < N_2 \quad \forall t \in [0,1], \tag{19}$$

then it solves also the original problem (1), (2).

We get the multiplicity result by combining the results of Proposition 3.1 and Proposition 3.2.

Proposition 3.3 *If the linear part in (18) is i-nonresonant with respect to the boundary conditions (2) and any solution $(x(t), y(t))$ of the problem (18), (2) satisfies the estimates (19), then the original problem (1), (2) has an i-type solution.*

Remark 3.1. Of course, the type of a solution $\xi(t)$ of the problem (1), (2) is induced by that of a solution $(\xi(t), \eta(t))$ of equivalent system (16) (Definition 2.2).

Definition 3.1 *We say that the problem (1), (2) allows for quasilinearization if it is possible to represent the equation (1) in the form (18) so that any solution of the quasi-linear problem (18),(2) satisfies the estimates (19). We say that two quasilinearizations are* ***essentially different*** *if the respective pairs (k,l) fall into different oscillation classes i. e. the products $k^2 l^2$ belong to different intervals of the form $(\pi^2 i^2, \pi^2 (i+1)^2)$.*

Proposition 3.4 *Suppose that m different quasilinearizations are possible for the problem (1), (2). Then this problem has at least m solutions of different types.*

The latter result is a direct consequence of propositions 3.1, 3.2, 3.3. Examples of application of Proposition 3.4 to the study of multiple solutions to some nonlinear problems are given in the last section.

3.3 Equation $x'' + f(t, x) = 0$

The problem

$$x'' + f(t, x) = 0, \quad x(0) = 0, \ x(1) = 0, \tag{20}$$

which is obtained from (1), (2) for $\Phi(x')x'$, was intensively studied by many authors. We mention only the results by Knobloch [7], Jackson and Schrader [6] and Erbe [5]. Their results when applied to the problem (20) say that the problem (20) in presence of the proper pair of the lower and upper functions α and β has a specific solution $z(t)$, which is such that the linear equation of variations

$$y'' + f_x(t, z(t))y = 0 \tag{21}$$

is disconjugate in the interval $[0, 1]$. Recall that a second order linear differential equation is said to be disconjugate in the interval $I = [0, 1]$ if no its solution has more than one zero in $(0, 1)$. The words "proper pair of the lower and upper functions α and β" mean that there exist functions $\alpha, \beta \in C^2([0, 1], R)$ such that:

$$\begin{aligned} &\alpha''(t) + f(t, \alpha(t)) \geq 0, \\ &\beta''(t) + f(t, \beta(t)) \leq 0, \\ &\alpha(t) \leq \beta(t), \forall t \in [0, 1]. \end{aligned} \tag{22}$$

This result is closely connected to the quasilinearization process based on Theorem 3.1 and in fact it motivated our research. Indeed, suppose that f in (20) is bounded, that is, the estimate

$$|f(t, x)| < M \quad \forall (t, x) \in [0, 1] \times R \tag{23}$$

is valid. Then proper α and β exist in the form $\alpha = M(t-0.5)^2 - M$, $\beta = -M(t-0.5)^2 + M$. Notice that $\alpha''(t) + f(t, \alpha(t)) = 2M + f(t, \alpha) > 0$, $\beta''(t) + f(t, \beta(t)) = -2M + f(t, \alpha) < 0$ and $\beta(t) \geq \frac{3}{4}M > -\frac{3}{4}M \geq \alpha(t)$ for any $t \in [0, 1]$. Accordingly to the above mentioned result there exists a solution $z(t)$ to the problem with the property that the linear equation (21) is disconjugate in $[0, 1]$. On the other hand, it follows from Theorem 3.1 that there must exist a 0-type solution $w(t)$ since the linear part in equation $x'' = 0$ or, equivalently, in system $x' - y = 0$, $y' = 0$, is 0-nonresonant with respect to the boundary conditions (2). Equation of variations $y'' + f_x(t, w(t))y = 0$, due to definition of 0-type solution, also is disconjugate in the interval $[0, 1]$.

In order to state results for the problem (20) we must adapt definitions of section 2 to our case.

Definition 3.2 *A linear part in the equation*

$$x'' + k^2 x = F(t, x) \tag{24}$$

is called i-nonresonant with respect to the boundary conditions (2) if $k^2 \in (i^2\pi^2, (i+1)^2\pi^2)$, $i \in \{0, 1, \ldots\}$.

Definition 3.3 *Let $\xi(t)$ be some solution of the BVP (20). We say that ξ is an i-type solution if for some small number $\varepsilon > 0$ the difference $u(t; \gamma)x(t; \gamma) - \xi(t)$ has exactly i zeros in the interval $(0, 1)$ and $u(1; \gamma) \neq 0$ for $\gamma \in (0, \varepsilon)$, where $x(t; \gamma)$ is a solution of the Cauchy problem*

$$x'' + f(t, x) = 0, \quad x(0) = 0, \ x'(0) = \xi'(0) + \gamma. \tag{25}$$

Consider modified equation

$$x'' + k^2 x = k^2 x - f(t,x) =: F(t,x)$$

and quasi-linear equation

$$x'' + k^2 x = \hat{F}(t,x) := F(t, \delta(-N, x, N)), \tag{26}$$

where $N > 0$ is some number.

Definition 3.4 *We say that the problem (20) allows for quasilinearization if it is possible to represent the equation in (20) in the form (26) so that any solution $x(t)$ of the quasi-linear problem (26),(2) satisfies the estimate $|x(t)| \leq N$. We say that two quasilinearizations are* ***essentially different*** *if the respective coefficients k fall into different oscillation classes i. e. the coefficients k^2 belong to different intervals of the form $(\pi^2 i^2, \pi^2 (i+1)^2)$.*

Proposition 3.5 *Suppose that m essentially different quasilinearizations are possible for the problem (20). Then this problem has at least m solutions of different types.*

Proposition 3.6 *Let $M_k(N) = \sup|\hat{F}(t,x)|$ and $\Gamma_k = \sup|G_k(t,x)|$, where G_k is the Green's function for the linear problem $x'' + k^2 x = 0,\ x(0) = 0,\ x(1) = 0$. If the inequality*

$$\Gamma_k \cdot M_k \leq N \tag{27}$$

holds then the problem (20) allows for quasilinearization and therefore has a solution of definite type.

Proof. The problem (26), (2) has a solution ([3]). Any solution $x(t)$ of this problem solves also the integral equation

$$x(t) = \int_0^1 G_k(t,s) \hat{F}(s, x(s))\, ds. \tag{28}$$

Then by virtue of (27)

$$|x(t)| \leq \Gamma_k \cdot M_k \leq N \quad \forall t \in [0,1]. \tag{29}$$

Since $\hat{F}(t,x) = F(t,x) = k^2 x - f(t,x)$ for $(t,x) \in [0,1] \times [-N,N]$, a solution $x(t)$ solves also the problem (20). $\square$

4 Application

Consider the differential equation

$$\frac{d}{dt}\Phi(x') + f(x) = 0, \tag{30}$$

where $t \in I := [0,1]$, $\Phi(z) := \mu^2 |z|^{\frac{1}{p}} \operatorname{sgn} z$, $f(x) := \nu^2 |x|^p \operatorname{sgn} x$, $\mu \neq 0$, $\nu \neq 0$, $p > 0$, $p \neq 1$, together with the boundary conditions (2).

Denote $\Phi(x') = y$, then obtain a two-dimensional differential system

$$\begin{cases} x' = \quad \Phi^{-1}(y), \\ y' = -f(x) \end{cases} \tag{31}$$

or

$$\begin{cases} x' = \quad |y|^p \operatorname{sgn} y, \\ y' = -\lambda^2 |x|^p \operatorname{sgn} x, \end{cases} \tag{32}$$

where $\lambda^2 = \dfrac{\nu^2}{\mu^2}$.

Theorem 4.1 *If there exists some $k \in \big(i\pi,\, (i+1)\pi\big)$, $i \in \{0,\, 1,\, 2,\, \ldots\}$, which satisfies the inequalities*

$$\frac{k}{|\sin k|} \cdot p^{\frac{p}{1-p}} \cdot |p-1| \cdot \big(1 + \lambda^{\frac{2}{1-p}}\big) < \gamma \qquad \textit{for} \quad \lambda^{\frac{2}{1-p}} \geq 1 \tag{33}$$

or

$$\frac{k}{|\sin k|} \cdot p^{\frac{p}{1-p}} \cdot |p-1| \cdot \big(1 + \lambda^{\frac{2}{p-1}}\big) < \gamma \qquad \textit{for} \quad \lambda^{\frac{2}{1-p}} < 1, \tag{34}$$

where γ is a root of the equation $\gamma^p = \gamma + (p-1) \cdot p^{\frac{p}{1-p}}$, then there exists an i-type solution of the nonlinear problem (30), (2).

Proof. Equation (30) reduces to the system (32) which, in turn, is equivalent to the system

$$\begin{cases} x' - k\,y = |y|^p \operatorname{sgn} y - k\,y, \\ y' + k\,x = k\,x - \lambda^2 |x|^p \operatorname{sgn} x, \end{cases} \tag{35}$$

where the coefficient $k > 0$ satisfies $\sin k \neq 0$.

Notice that a linear part in the system (35) is non-resonant with respect to the given boundary conditions (2), if the coefficient k satisfies $\sin k \neq 0$;
if $k \in (i\pi,\, (i+1)\pi)$, $i \in \{0,\, 1,\, 2,\, \ldots\}$, then the respective linear part is i-nonresonant with respect to the boundary conditions (2).

We wish to make bounded the right sides in the system (35). Denote

$$U_k(y) := |y|^p \operatorname{sgn} y - k\,y.$$

We calculate the value of this function at the point of local extremum y_0. Set

$$M_y = |U_k(y_0)| = \left(\frac{k}{p}\right)^{\frac{p}{p-1}} |p-1|. \tag{36}$$

Choose N_y such that

$$|y| \leq N_y \Rightarrow |U_k(y)| \leq M_y.$$

Computation gives that

$$N_y = k^{\frac{1}{p-1}} \gamma, \tag{37}$$

where a constant γ ($\gamma > 1$) is a root of the equation $\gamma^p = \gamma + (p-1)p^{\frac{p}{1-p}}$.

Denote

$$V_k(x) := k\,x - |x|^p \operatorname{sgn} x.$$

We calculate the value of this function at the point of local extremum x_0. Set

$$M_x = |V_k(x_0)| = \lambda^{\frac{2}{1-p}} \cdot \left(\frac{k}{p}\right)^{\frac{p}{p-1}} \cdot |p-1|. \tag{38}$$

Choose N_x such that

$$|x| \le N_x \Rightarrow |V_k(x)| \le M_x.$$

Computation gives that

$$N_x = \left(\frac{k}{\lambda^2}\right)^{\frac{1}{p-1}} \cdot \gamma. \tag{39}$$

Instead of the functions $U_k(y)$, $V_k(x)$ consider

$$\hat{U}_k(y) := U_k(\delta(-N_y, y, N_y)),$$
$$\hat{V}_k(x) := V_k(\delta(-N_x, x, N_x)).$$

Denote

$$\sup|\hat{U}_k(y)| = M_y, \quad \sup|\hat{V}_k(x)| = M_x.$$

The nonlinear system (32) and the quasilinear one

$$\begin{cases} x' - k\,y = \hat{U}_k(y), \\ y' + k\,x = \hat{V}_k(x) \end{cases} \tag{40}$$

are equivalent in a domain

$$\Omega_k = \{(t,\,x,\,y) :\ 0 \le t \le 1,\ |x(t)| \le N_x,\ |y(t)| \le N_y\}. \tag{41}$$

(In the case of $0 < p < 1$ we suppose that $(t,\,0,\,0) \notin \Omega_k$ in order to exclude the trivial solution of indefinite type).

The quasi-linear problem (40), (2) is solvable if k is such that homogeneous problem

$$\begin{cases} x' - k\,y = 0, \\ y' + k\,x = 0, \\ x(0) = 0,\ x(1) = 0 \end{cases} \tag{42}$$

has only the trivial solution. The respective solution $X_k(t) := \big(x_k(t),\, y_k(t)\big)$ can be written in the integral form

$$\begin{cases} x_k(t) = \displaystyle\int_0^1 \Big(G_k^{11}(t,s)\,U_k\big(y(s)\big) + G_k^{12}(t,s)\,V_k\big(x(s)\big)\Big)\,ds, \\ y_k(t) = \displaystyle\int_0^1 \Big(G_k^{21}(t,s)\,U_k\big(y(s)\big) + G_k^{22}(t,s)\,v_k\big(x(s)\big)\Big)\,ds, \end{cases} \tag{43}$$

where $G_k^{ij}(t,\, s)$ $(i, j = 1, 2)$ are the elements of the Green's matrix to the respective homogeneous problem (42).

Then

$$\begin{cases} |x_k(t)| \le \Gamma_{11}(k) \cdot M_y + \Gamma_{12}(k) \cdot M_x, \\ |y_k(t)| \le \Gamma_{21}(k) \cdot M_y + \Gamma_{22}(k) \cdot M_x, \end{cases} \tag{44}$$

where $\Gamma_{ij}(k)$ $(i, j = 1, 2)$ are the estimates of the respective elements $G_k^{ij}(t,\, s)$ of the Green's matrix. If the inequalities

$$\begin{array}{l} \Gamma_{11}(k) \cdot M_y + \Gamma_{12}(k) \cdot M_x \,<\, N_y, \\ \Gamma_{21}(k) \cdot M_y + \Gamma_{22}(k) \cdot M_x \,<\, N_x \end{array} \tag{45}$$

hold then the nonlinear problem (32), (2) (or, equivalently, the original problem (30), (2)) allows for quasilinearization and therefore has a solution of definite type.

Since the Green's matrix of the homogeneous linear problem (42) is given by

$$\mathbb{G}_k(t,\, s) = \begin{cases} \dfrac{1}{\sin k} \begin{pmatrix} -\cos(ks)\sin(k(t-1)) & \sin(ks)\sin(k(t-1)) \\ -\cos(ks)\cos(k(t-1)) & \sin(ks)\cos(k(t-1)) \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 0 \le s \le t \le 1, \\ \dfrac{1}{\sin k} \begin{pmatrix} -\sin(kt)\cos(k(s-1)) & \sin(kt)\sin(k(s-1)) \\ -\cos(kt)\cos(k(s-1)) & \cos(kt)\sin(k(s-1)) \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 0 \le t < s \le 1, \end{cases} \tag{46}$$

therefore

$$\left| G_k^{ij}(t,\, s) \right| \le \frac{1}{|\sin k|} =: \Gamma_k, \qquad (i, j = 1, 2). \tag{47}$$

Taking into consideration the expressions for M_y (36), M_x (38), N_y (37), N_x (39), and the estimate Γ_k (47) we obtain that inequalities in (45) take a form

$$\frac{1}{|\sin k|} \cdot \left(\frac{k}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \left(1 + \lambda^{\frac{2}{1-p}}\right) < \left(\frac{k}{\lambda^2}\right)^{\frac{1}{p-1}} \cdot \gamma \tag{48}$$

and

$$\frac{1}{|\sin k|} \cdot \left(\frac{k}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \left(1 + \lambda^{\frac{2}{1-p}}\right) < k^{\frac{1}{p-1}} \cdot \gamma. \tag{49}$$

If $\lambda^{\frac{2}{1-p}} \ge 1$, then both inequalities (48) and (49) are valid if the inequality (33) is fulfilled.

If $\lambda^{\frac{2}{1-p}} < 1$, then the inequalities (48) and (49) hold if the inequality (34) is fulfilled.

Thus a fulfilment of the inequalities (33) or (34) is a sufficient condition for the solvability to the nonlinear problem (30), (2). If the inequalities above fulfill for $k \in \big(i\pi,\, (i+1)\pi\big)$, $i \in \{0,\, 1,\, 2,\, \ldots\}$, then the linear part in the system (35) is i-nonresonant with respect to the given boundary conditions (2), therefore the nonlinear problem (30), (2) has an i-type solution.□

Corollary 4.1 *If there exist numbers* k_j*,* $j \in \{0, 1, 2, \ldots, n\}$*, such that* $k_j \in \big(j\pi, (j+1)\pi\big)$ *and the inequalities* (33) *or* (34) *are satisfied, then there exist at least* $n+1$ *solutions of different types to the nonlinear problem* (30), (2).

For k of the form $k = \frac{\pi}{2} + \pi i$, $i \in \{0, 1, 2, \ldots\}$, the basic inequalities (33), (34) to be verified take the form

$$k \cdot p^{\frac{p}{1-p}} \cdot |p-1| \cdot \big(1 + \lambda^{\frac{2}{1-p}}\big) < \gamma \qquad \text{for} \quad \lambda^{\frac{2}{1-p}} \geq 1 \tag{50}$$

or

$$k \cdot p^{\frac{p}{1-p}} \cdot |p-1| \cdot \big(1 + \lambda^{\frac{2}{p-1}}\big) < \gamma \qquad \text{for} \quad \lambda^{\frac{2}{1-p}} < 1. \tag{51}$$

Table 1. Case $p > 1$.

p	γ	λ^2	k_i
$\frac{4}{3}$	1.27033	$\frac{7}{8} \le \lambda^2 \le \frac{8}{7}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
$\frac{5}{4}$	1.28132	$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
$\frac{6}{5}$	1.28840	$\frac{4}{5} \le \lambda^2 \le \frac{5}{4}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{13}{14} \le \lambda^2 \le \frac{14}{13}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
$\frac{7}{6}$	1.29334	$\frac{4}{5} \le \lambda^2 \le \frac{5}{4}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{9}{10} \le \lambda^2 \le \frac{10}{9}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
		$\frac{82}{83} \le \lambda^2 \le \frac{83}{82}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2}$
$\frac{8}{7}$	1.29698	$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{8}{9} \le \lambda^2 \le \frac{9}{8}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
		$\frac{21}{22} \le \lambda^2 \le \frac{22}{21}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2}$
$\frac{9}{8}$	1.29978	$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{8}{9} \le \lambda^2 \le \frac{9}{8}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
		$\frac{15}{16} \le \lambda^2 \le \frac{16}{15}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2}$
		$\frac{69}{70} \le \lambda^2 \le \frac{70}{69}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2};\ k_4 = \frac{9\pi}{2}$

Table 2. Case $0 < p < 1$.

p	γ	λ^2	k_i
$\frac{3}{4}$	1.37580	$\frac{2}{3} \le \lambda^2 \le \frac{3}{2}$	$k_0 = \frac{\pi}{2}$
		$\frac{7}{8} \le \lambda^2 \le \frac{8}{7}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
$\frac{4}{5}$	1.36324	$\frac{2}{3} \le \lambda^2 \le \frac{3}{2}$	$k_0 = \frac{\pi}{2}$
		$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{45}{46} \le \lambda^2 \le \frac{46}{45}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
$\frac{5}{6}$	1.35538	$\frac{2}{3} \le \lambda^2 \le \frac{3}{2}$	$k_0 = \frac{\pi}{2}$
		$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{13}{14} \le \lambda^2 \le \frac{14}{13}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
$\frac{6}{7}$	1.34999	$\frac{3}{4} \le \lambda^2 \le \frac{4}{3}$	$k_0 = \frac{\pi}{2}$
		$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{10}{11} \le \lambda^2 \le \frac{11}{10}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
		$\frac{45}{46} \le \lambda^2 \le \frac{46}{45}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2}$
$\frac{7}{8}$	1.34607	$\frac{3}{4} \le \lambda^2 \le \frac{4}{3}$	$k_0 = \frac{\pi}{2}$
		$\frac{5}{6} \le \lambda^2 \le \frac{6}{5}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2}$
		$\frac{9}{10} \le \lambda^2 \le \frac{10}{9}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2}$
		$\frac{20}{21} \le \lambda^2 \le \frac{21}{20}$	$k_0 = \frac{\pi}{2};\ k_1 = \frac{3\pi}{2};\ k_2 = \frac{5\pi}{2};\ k_3 = \frac{7\pi}{2}$

In the Table 1 and Table 2 above the results of calculations are summarized. For certain values of p and λ^2 the values of k in the form $k_i = \frac{\pi}{2} + \pi i$, $i \in \{0, 1, 2, \ldots\}$ are given, which satisfy the inequalities (50) and (51). A lower index i of a number k_i in these tables testifies about an existence of an i-type solution to the problem (32), (2). So these tables may be interpreted as a set of multiplicity results for the modified problem (32), (2) and original problem (30), (2).

Notice that in the case of $p > 1$ the trivial solution of the modified problem (32), (2) is a 0-type solution; in a case, when $0 < p < 1$, a certain nontrivial solution of the problem (32), (2) is a 0-type solution. Thus both tables for certain values of p and λ^2 point to existence of nontrivial solutions to the original problem (30), (2).

5 Example

Consider the problem

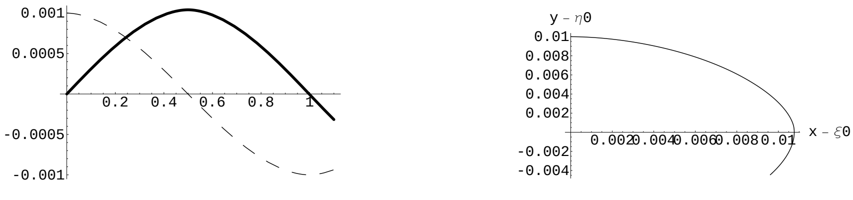
$$\begin{cases} \dfrac{d}{dt}\big(14|x'|^{\frac{6}{5}}\operatorname{sgn} x'\big) + 13|x|^{\frac{5}{6}}\operatorname{sgn} x = 0, \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{52}$$

which is a special case of the problem (30), (2) with $p = \dfrac{5}{6}$, $\mu^2 = 14$ and $\nu^2 = 13$. The given problem (52) is equivalent to the problem

$$\begin{cases} x' = |y|^{\frac{5}{6}}\operatorname{sgn} y, \\ y' = -\dfrac{13}{14}|x|^{\frac{5}{6}}\operatorname{sgn} x, \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{53}$$

In accordance with calculations (see Table 2) and Corollary 4.1 there exist at least three solutions of different types to the problem (53). We have computed them.

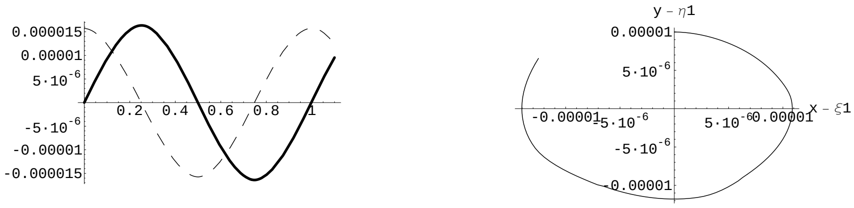
The solid line in Figure 5.1a describes a solution of the problem (52), the solid and dashed lines together indicate a solution $\big(\xi_0(t),\, \eta_0(t)\big)$ of the modified problem (53). This solution $\big(\xi_0(t),\, \eta_0(t)\big)$ is a 0-type solution because the angular function $\Theta(t;\delta)$ of the difference between neighboring solution $\big(x(t;\delta),\, y(t;\delta)\big)$ and $\big(\xi_0(t),\, \eta_0(t)\big)$, defined by the initial condition $\Theta(0;\delta) = 0$, for any $\delta > 0$ does not take values of the form πn, $n \in \mathbb{N}$ in the interval $t \in (0, 1]$. A phase portrait of the above difference with $\delta = 1 \cdot 10^{-2}$ is depicted in Figure 5.1b. The initial data of the 0-type solution are $\xi_0(0) = 0$, $\xi_0'(0) = 3162.28 \cdot 10^{-6}$.



a b

Figure 5.1 0-type solution of the problem (53).

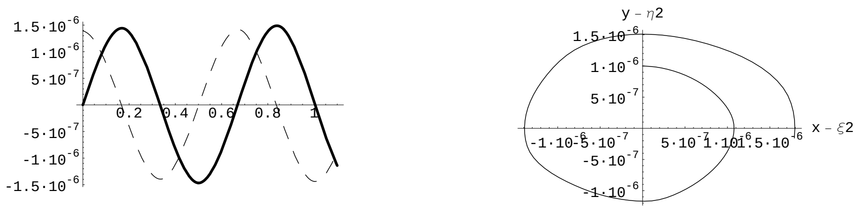
The solid line in Figure 5.2a describes another solution of problem (52), together with the dashed line they indicate a solution $\big(\xi_1(t),\, \eta_1(t)\big)$ of the problem (53). The solution $\big(\xi_1(t),\, \eta_1(t)\big)$ is an 1-type solution because the angular function $\Theta(t;\delta)$ of the difference between the respective neighboring solution $\big(x(t;\delta),\, y(t;\delta)\big)$ and $\big(\xi_1(t),\, \eta_1(t)\big)$, defined by same initial condition $\Theta(0;\delta) = 0$, for any $\delta \in (0,\, 0.0009]$ takes exactly one value of the form $\pi n,\ n \in \mathbb{N}$ in the interval $t \in (0,\, 1]$. A phase portrait of the above difference with $\delta = 1 \cdot 10^{-5}$ is depicted in Figure 5.2b. The initial data of the 1-type solution is given by $\xi_1(0) = 0, \quad \xi_1'(0) = 99.2916 \cdot 10^{-6}$.



a b

Figure 5.2 1-type solution of the problem (53).

Figure 5.3a shows the third solution of the given problem (52) (in solid). Both lines in Figure 5.3a (solid and dashed) indicate a 2-type solution of the problem (53), because the respective angular function $\Theta(t;\delta)$ for any $\delta \in (0,\, 0.000014]$ in the interval $t \in (0,\, 1]$ takes exactly two values of the form $\pi n,\ n \in \mathbb{N}$ (see Figure 5.3b for $\delta = 1 \cdot 10^{-6}$). The initial data of the 2-type solution is given by $\xi_2(0) = 0, \quad \xi_2'(0) = 13.1577 \cdot 10^{-6}$.



a b

Figure 5.3 2-type solution of the problem (53).

Along with the solutions mentioned above there exists a trivial solution of the given problem (52) (as well there exists a trivial solution $\big(\xi_*(t), \eta_*(t)\big)$ of the modified problem (53), where $\xi_*(t) \equiv 0$, $\eta_*(t) \equiv 0$), but by introduced terminology a type of a trivial solution isn't determined (it is a case $0 < p < 1$).

References

[1] A. Cabada and R.L. Pouso: *Extremal Solutions of Strongly Nonlinear Discontinuous Second Order Equations with Nonlinear Functional Boundary Conditions.* Nonlinear Anal., 42 (2000), 1377 – 1396.

[2] M. Cherpion, C. de Coster and P. Habets: *Monotone iterative methods for boundary value problems.* Differential Integral Equations. 12 (3), 1999, 309 – 338.

[3] R. Conti. Equazioni differenziali ordinarie quasilineari con condizioni lineari. - Ann. mat. pura ed appl., **57** (1962), 48 – 62.

[4] Y. Eidelman, V. Milman and A. Tsolomitis. Functional Analysis: An Introduction, AMS, 2004.

[5] L.H. Erbe. Nonlinear Boundary Value Problems for Second Order Differential Equations. - J. Diff. Equations, **7** (1970), pp. 459 – 472.

[6] L.K. Jackson and K.W. Schrader. Comparison theorems for nonlinear differential equations. - J. Diff. Equations, **3** (1967), pp. 248 – 255.

[7] H.W. Knobloch. Comparison theorems for nonlinear second order differential equations. J. Diff. Equations, **1** (1965), 1 – 26.

[8] I. Kiguradze. Initial and boundary value problems for systems of ordinary differential equations (Vol.1 Linear theory). - Metsniereba-Publishing House, Tbilissi, 1997 (in Russian).

[9] I. Yermachenko and F. Sadyrbaev. Types of solutions and multiplicity results for two-point nonlinear boundary value problems. - Elsevier Ltd., Nonlinear Analysis **63** (2005), pp. e1725 – e1735. (electr.version http://www.elsevier.com/locate/na)

[10] I. Yermachenko and F. Sadyrbaev. Quasilinearization and multiple solutions of the Emden-Fowler type equation. - Mathematical Modeling and Analysis, vol. 10, **1** (2005), pp. 41 – 50 .

[11] I. Yermachenko and F. Sadyrbaev. Multiplicity results for two-point nonlinear boundary value problems. Proc. of the conference CDDEA, Zhilina, Slovakia, June 2006 (to appear).

И. Ермаченко, Ф. Садырбаев. Результаты о числе решений краевой задачи для уравнения с Ф-Лапласианом.

Аннотация. Рассматривается уравнение типа $\frac{d}{dt}\Phi(x') + f(t,x) = 0$ с краевыми условиями Дирихле. Это уравнение сводится к двумерной дифференциальной системе. Затем т.н. процесс квазилинеаризации применяется для доказательства (при определенных условиях) существования многих решений задачи.

УДК 517.927

I. Jermačenko, F. Sadirbajevs. Par robežproblēmas Φ-Laplasian tipa diferenciālvienādojumam neunitāti.

Anotācija. Tiek apskatīts diferenciālvienādojums $\frac{d}{dt}\Phi(x')+f(t,x)=0$ kopā ar Dirihlē robežnosacījumiem. Šis vienādojums tiek reducēts uz divu dimensiju diferenciālsistēmu. Kvazilinearizācijas metode tiek pielietota, lai noteiktu dažādu tipu atrisinājumu eksistences nosacījumus.

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