Nonlinear spectra for two-parameter eigenvalue problems

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Summary. Eigenvalue problems of the form $x'' = -\lambda f(x) + \mu g(x)$, (i), $x(0) = 0$, $x(1) = 0$, (ii) are considered. We are looking for (λ, μ) such that the problem $(i), (ii)$ has a nontrivial solution. This problem generalizes the famous Fučik problem for piece-wise linear equations. In order to show that nonlinear Fučik spectra may differ essentially from the classical ones, we consider functions $f(x)$ and $g(x)$ such that they are piece-wise linear and the first zero functions t_1 and τ_1 can be computed explicitly. Then it is possible to construct explicitly the respective Fučik like spectra.

MSC: 34B15
Keywords: nonlinear spectra, jumping nonlinearity, asymptotically asymmetric nonlinearities, Fuchik spectrum.

1 Introduction

In this paper we consider boundary value problems of the form

$$x'' = -\lambda f(x) + \mu g(x), \tag{1}$$

$$x(0) = 0, \; x(1) = 0, \tag{2}$$

where λ and μ are non-negative parameters and f and g are continuous (piece-wise linear) functions such that $f(x) > 0$ for $x > 0$ and $f = 0$ for $x < 0$ and, respectively, $g(x) > 0$ for $x < 0$ and $g = 0$ for $x > 0$. This problem can be written in a usual form

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(x), & \text{if} \quad x < 0. \end{cases} \tag{3}$$

Any nontrivial solution $x(t)$ of equation (1) (or, which is the same, of (3)) satisfies the condition $x(t)x''(t) \leq 0$ for any t. Therefore behavior of solutions is rather oscillatory.

In this research we continue the study of nonlinear Fučik type spectra, which was initiated in [8]. At the very beginning of the story stands the famous Fučik equation

$$x'' = -\lambda x^+ + \mu x^-, \tag{4}$$

where $x^+ = \max\{x, 0\}, \; x^- = \max\{-x, 0\}$.

This equation may be written also as

$$x'' = \begin{cases} -\lambda x, & \text{if} \quad x \geq 0 \\ -\mu x, & \text{if} \quad x < 0. \end{cases} \tag{5}$$

Equation (4) contains a piece-wise linear function in the right side which possesses some important properties of the linear functions. For example, the positive homogeneity property holds, that is $F(\alpha x) = \alpha F(x),\ \alpha > 0$, where $F(x)$ stands for the right side in (4). Formally equation (4) is nonlinear and the additivity property fails to hold, that is the sum of two solutions $x_1(t)$ and $x_2(t)$ of (4) need not to be a solution. It was the idea of Fuchik [1] to modify a linear equation in this way and to consider nonlinear ("almost" linear) equations of the form (4).

The Fuchik spectrum for the problem (4), (2) is defined as a set of all pairs (λ, μ), for which the problem has a nontrivial solution. This spectrum is well known ([5, § 35]) and is depicted in Fig. 2.

The Fučik spectrum was intensively studied after the pioneering works by Fučik for various equations and boundary conditions (some references can be found in [3]).

It is to be mentioned that these studies are useful when investigating the so called asymptotically asymmetric nonlinearities, jumping nonlinearities and even practical problems in engineering. For the whole story one may consider references in [8]. The goal of this paper is very moderate.

We recall first the result about Fučik type spectra for equations of the type (1). Then we consider piece-wise linear functions f and g and investigate the first zero functions. The explicit formulas for the first zeros functions are obtained and the respective Fučik type spectra are constructed.

Discussion on differences comparing with the classical Fučik spectra follows.

2 One parameter problems

Let us recall some facts about the nonlinear eigenvalue problem

$$x'' = -\lambda f(x), \quad x(0) = 0,\ x(1) = 0. \tag{6}$$

One has to consider problems of this type when looking for positive solutions of (1), (2) (respectively, the problem $x'' = \mu g(x), \quad x(0) = 0,\ x(1) = 0$ should be considered when looking for negative solutions of (1), (2)).

The problem (6) was studied in numerous papers, see [6], [4], for instance. It is known that any positive solution $x(t)$ of (6) is symmetric with respect to the middle point $t = \frac{1}{2}$, where the maximal value is attained.

We assume that $f(x)$ satisfies the following condition:
(A1) A first zero $t_1(\alpha)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(0) = 0,\ u'(0) = \alpha \tag{7}$$

exists for any $\alpha > 0$.

Similar property can be assigned to $g(x)$.

We assume that $g(x)$ satisfies the condition:
(A2) A first zero $\tau_1(\beta)$ of a solution to the Cauchy problem

$$v'' = g(v), \quad v(0) = 0,\ v'(0) = -\beta \tag{8}$$

exists for any $\beta > 0$.

Simple examples of $f(x)$ possessing the property **(A1)** are the functions $f(x) = x^3$ ($t_1(\alpha)$ decreases from $+\infty$ to zero as α increases from zero to $+\infty$) and $f(x) = x^{\frac{1}{3}}$ ($t_1(\alpha)$ increases from zero to $+\infty$ as α increases from zero to $+\infty$). This can be verified by direct calculation.

Proposition 2.1 *Suppose that $f(x)$ satisfies the condition* **(A1)** *and $t_1(\alpha)$ maps $(0, +\infty)$ onto $(0, +\infty)$ continuously. Then the problem (6) has a continuous spectrum.*

Proof. Fix $\lambda > 0$ and consider a solution $u(t;\alpha)$ of the Cauchy problem (7). This solution has its first positive zero at $t_1(\alpha)$. Consider a function $X(t) := u(\sqrt{\lambda}\,t;\alpha)$. This function solves the equation in (6). Moreover, $X(0) = 0$ and $X(\frac{t_1(\alpha)}{\sqrt{\lambda}}) = 0$. In view of properties of the function $t_1(\alpha)$ for fixed λ a value $\alpha_0 > 0$ exists such that $\frac{t_1(\alpha_0)}{\sqrt{\lambda}} = 1$. ■

The value $\max_{[0,1]} x(t) := \|x\|$ and λ relate as

$$\|x\| \cdot \lambda = 2\sqrt{2} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

The problem has continuous spectrum therefore, that is, for any positive λ there exists a unique positive solution of the problem.

Similarly the problem

$$x'' = \mu g(x), \quad x(0) = 0,\ x(1) = 0, \quad x(t) < 0 \text{ in } (0,1) \tag{9}$$

also has continuous spectrum.

A solution of the problem (6) under the condition **(A1)** (and (9) under the condition **(A2)**) is unique, however, if the normalization condition $x'(0) = 1$ (resp.: $x'(0) = -1$) is imposed.

3 Basic formulas

Consider

$$x'' = \begin{cases} -\lambda f(x), & \text{if } \ x \geq 0 \\ \mu g(x), & \text{if } \ x < 0, \end{cases} \qquad x(0) = x(1) = 0, \tag{10}$$

where $f(x)$ and $g(x)$ are positive valued continuous functions described in Introduction. Suppose that f and g satisfy the conditions **(A1)** and **(A2)** respectively.

It can be shown easily that this problem has continuous spectrum ([8]).

One is led thus to the conclusion that in order to have reasonable nonlinear eigenvalue problems solutions under some normalization should be considered.

Consider

$$x'' = \begin{cases} -\lambda f(x), & \text{if } \ x \geq 0 \\ \mu g(x), & \text{if } \ x < 0, \end{cases} \qquad x(0) = x(1) = 0, \quad |x'(0)| = 1. \tag{11}$$

Let us recall the main result in [8].

Theorem 3.1 *Let the conditions* **(A1)** *and* **(A2)** *hold with the functions* $t_1(\gamma)$ *and* $\tau_1(\delta)$. *The Fuchik spectrum for the problem (11) is given by the relations* ($i = 1, 2, \ldots$)*:*

$$F_0^+ = \Big\{ \big(\lambda, \mu\big) : \ \lambda \textit{ is a solution of } \ \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1, \quad \mu \geq 0 \Big\}, \tag{12}$$

$$F_0^- = \Big\{ \big(\lambda, \mu\big) : \lambda \geq 0, \ \mu \textit{ is a solution of } \ \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1 \Big\}, \tag{13}$$

$$F_{2i-1}^+ = \Big\{ (\lambda; \mu) : \quad i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1 \Big\}, \tag{14}$$

$$F_{2i-1}^- = \Big\{ (\lambda; \mu) : \quad i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1 \Big\}, \tag{15}$$

$$F_{2i}^+ = \Big\{ (\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1 \Big\}, \tag{16}$$

$$F_{2i}^- = \Big\{ (\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1 \Big\}. \tag{17}$$

4 Piece-wise linear functions f and g. Definitions

Let

$$0 < a_1 < a_2 < a_3, \quad b_1 > b_2 > 0, \quad b_3 > b_2.$$

Consider a piece-wise linear function:

$$f(x) = \begin{cases} f_1(x), & 0 \leq x \leq a_1, \\ f_2(x), & a_1 \leq x \leq a_2, \\ f_3(x), & x \geq a_3, \end{cases} \tag{18}$$

$$f_1(x) = p_1 x + q_1, \quad f_2(x) = p_2 x + q_2, \quad f_3(x) = p_3 x + q_3,$$

$$f_1(0) = 0, \quad f_1(a_1) = f_2(a_1), \quad f_2(a_2) = f_3(a_2), \quad f_3(a_3) = b_3.$$

Notice that

$$\begin{aligned} p_1 &= \frac{b_1}{a_1}, & q_1 &= 0, \\ p_2 &= \frac{b_2 - b_1}{a_2 - a_1}, & q_2 &= \frac{b_1 a_2 - a_1 b_2}{a_2 - a_1}, \\ p_3 &= \frac{b_3 - b_2}{a_3 - a_2}, & q_3 &= \frac{b_2 a_3 - a_2 b_3}{a_3 - a_2}. \end{aligned}$$

4.1 The first zero functions t_1 and τ_1. Formulas

Consider the initial value problem

$$x'' = -\lambda f(x), \quad x(0) = 0, \quad x'(0) = \alpha > 0. \tag{19}$$

Let $x(t)$ stand for a solution of the problem (19) for $\lambda = 1$. Then the first positive zero of the $x(t)$ is given by

$$t_1(\alpha) = \int_0^{x_\alpha} \frac{2ds}{\sqrt{\alpha^2 - 2F(s)}}, \tag{20}$$

where $F(x) = \int_0^x f(s)ds$, but x_α ir the unique positive zero of the equation $\alpha^2 - 2F(x) = 0$.

One has by direct calculations that

1. if $0 \le \alpha \le \sqrt{2F(a_1)}$, then

$$t_1(\alpha) = \pi \sqrt{\frac{a_1}{b_1}},$$

2. if $\sqrt{2F(a_1)} \le \alpha \le \sqrt{2F(a_2)}$, then

$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \frac{D_2(\alpha)}{\left(-2b_1 + 2\sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1}\right)^2},$$

3. if $\alpha \ge \sqrt{2F_2(a_2)}$, then

$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \sqrt{\frac{a_3 - a_2}{b_3 - b_2}} \left[\pi - 2\arcsin \frac{2b_2}{\sqrt{D_3(\alpha)}}\right] +$$
$$+ 2\sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \left| \frac{-b_2 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1 - (a_2 - a_1)(b_1 + b_2)}}{-b_1 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1}} \right|,$$

where

$$D_2(\alpha) = 4\,\frac{b_1 - b_2}{a_1 - a_2}\alpha^2 + 4b_1\,\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2},$$

$$D_3(\alpha) = 4\,\frac{b_2 - b_3}{a_2 - a_3}\alpha^2 + 4\,\frac{-a_2 b_1 b_2 + a_1 b_2^2 + a_3 b_2^2 + a_2 b_1 b_3 - a_1 b_2 b_3 + a_2 b_2 b_3}{a_2 - a_3}.$$

Remark 4.1. The first zero function is asymptotically linear:

$$\lim_{\alpha \to +\infty} t_1(\alpha) = \sqrt{\frac{a_3 - a_2}{b_3 - b_2}}\pi.$$

Details of calculations can be found in Appendix.

5 The Fučik type spectra

We introduce new variables $\gamma = \frac{1}{\sqrt{\lambda}}$, $\delta = \frac{1}{\sqrt{\mu}}$. The advantage is that all the branches of the spectra but two are located in a bounded domain of the (γ, δ)-plane. One has that

- branches which describe solutions of the problem (4), (2) with at least one zero in $(0,1)$ transform to open intervals in the (γ,δ)-plane;
- the branch which describes a non-vanishing negative valued in $(0,1)$ solution of the problem (4), (2) is a horizontal ray with attached point at infinity $\left(\infty,\frac{1}{\pi}\right)$ (this point is an image of the point $(0,\pi^2)$ in the (λ,μ)-plane);
- the branch which describes a non-vanishing positive valued in $(0,1)$ solution of the problem (4), (2) is a vertical ray with attached point at infinity $\left(\frac{1}{\pi},\infty\right)$ (this point is an image of the point $(\pi^2,0)$ in the (λ,μ)-plane).

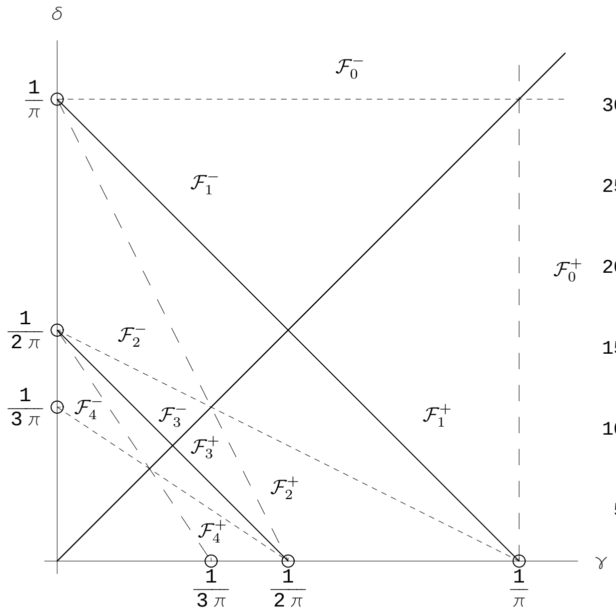

Figure 1: Fučik spectrum in the (γ,δ)-plane.

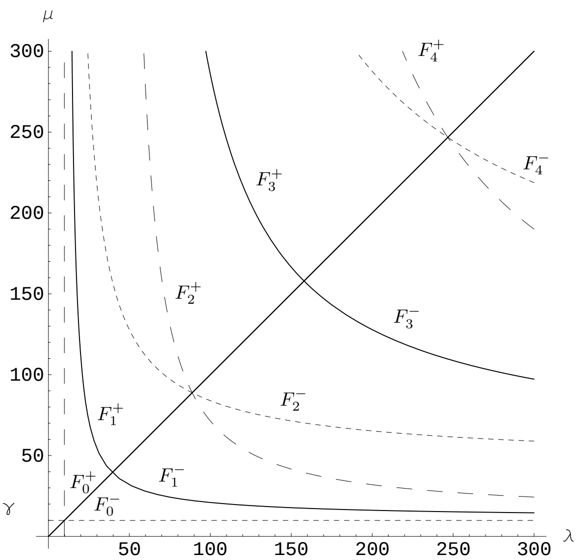

Figure 2: Fučik spectrum in the (λ,μ)-plane.

Theorem 3.1 in new variables takes the form:

Theorem 5.1 *Let the conditions* **(A1)** *and* **(A2)** *hold with the functions* $t_1(\gamma)$ *and* $\tau_1(\delta)$. *The Fuchik spectrum for the problem (11) in the plane* (γ,δ) *is given by the relations* $(i=1,2,\ldots)$*:*

$$\mathcal{F}_0^+ = \big\{(\gamma,\delta):\ \gamma \textit{ is a solution of } \gamma\, t_1(\gamma)=1,\quad \delta>0\big\}\cup \cup\big\{(\gamma,\infty):\ \gamma \textit{ is a solution of } \gamma\, t_1(\gamma)=1\big\}, \tag{21}$$

$$\mathcal{F}_0^- = \big\{(\gamma,\delta):\gamma>0,\ \delta \textit{ is a solution of } \delta\,\tau_1(\delta)=1\big\}\cup \cup\big\{(\infty,\delta):\ \delta \textit{ is a solution of } \delta\,\tau_1(\delta)=1\big\}, \tag{22}$$

$$\mathcal{F}_{2i-1}^+ = \big\{(\gamma;\delta):\quad i\gamma\, t_1(\gamma)+i\delta\,\tau_1(\delta)=1,\ \gamma>0,\ \delta>0\big\}, \tag{23}$$

$$\mathcal{F}_{2i-1}^- = \big\{(\gamma;\delta):\quad i\delta\,\tau_1(\delta)+i\gamma\, t_1(\gamma)=1,\ \gamma>0,\ \delta>0\big\}, \tag{24}$$

$$\mathcal{F}_{2i}^+ = \big\{(\gamma;\delta):\quad (i+1)\gamma\, t_1(\gamma)+i\delta\,\tau_1(\delta)=1,\ \gamma>0,\ \delta>0\big\}, \tag{25}$$

$$\mathcal{F}_{2i}^- = \big\{(\gamma;\delta):\quad (i+1)\delta\,\tau_1(\delta)+i\gamma\, t_1(\gamma)=1,\ \gamma>0,\ \delta>0\big\}. \tag{26}$$

Consider now the problem (11) with $g(x)=f(-x),\ x<0$. Then $\tau_1=t_1$.

Three examples follow for various piece-wise linear functions $f(x)$, where the numbers of roots of the equation $\gamma\, t_1(\gamma) = 1$ are different.

5.1 First example

Let parameters of the piece-wise linear function $f(x)$ in (18) be

$$a_1 = \frac{1}{500}, \qquad a_2 = 2, \qquad a_3 = 5,$$
$$b_1 = 200, \qquad b_2 = \frac{1}{10}, \qquad b_3 = 1800.$$

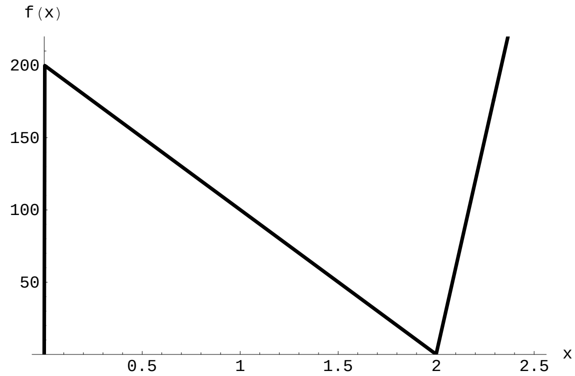

Figure 3: The graph of $y = f(x)$.

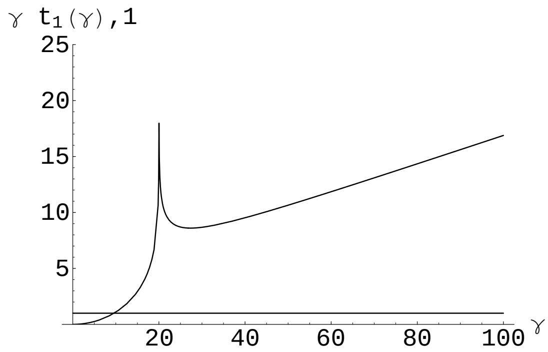

Figure 4: Graphs of $y = \gamma t_1(\gamma)$ and $y = 1$.

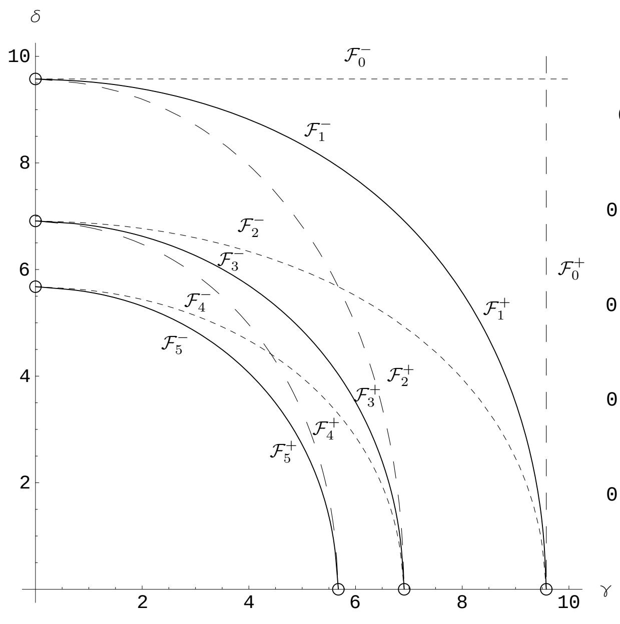

Figure 5: Fučik type spectrum in the (γ, δ)-plane.

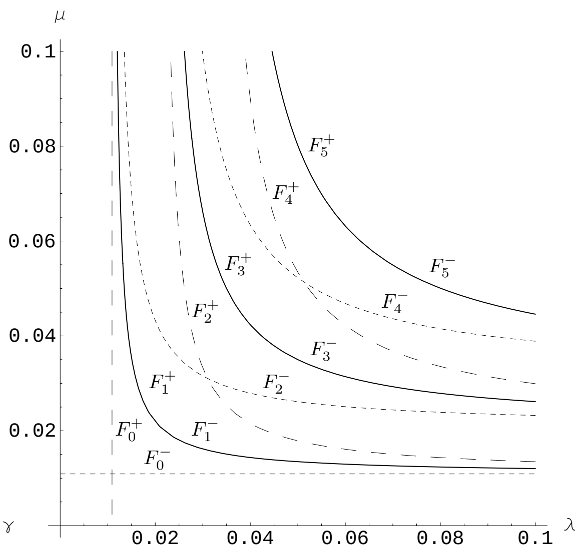

Figure 6: Fučik type spectrum in the (λ, μ)-plane.

Since the function $y = \gamma t_1(\gamma)$ is monotone beneath the line $y = 1$ as is seen in Fig. 4, equation

$$\gamma t_1(\gamma) + \delta \tau_1(\delta) = 1 \tag{27}$$

define monotone curves in the (γ, δ)-plane and branches of the spectrum look like those of the classical Fučik spectrum.

5.2 Second example

Let parameters of the piece-wise linear function $f(x)$ in (18) be

$$a_1 = 0.1, \qquad a_2 = 0.3, \qquad a_3 = 0.31,$$
$$b_1 = 9, \qquad b_2 = 0.5, \qquad b_3 = 150.$$

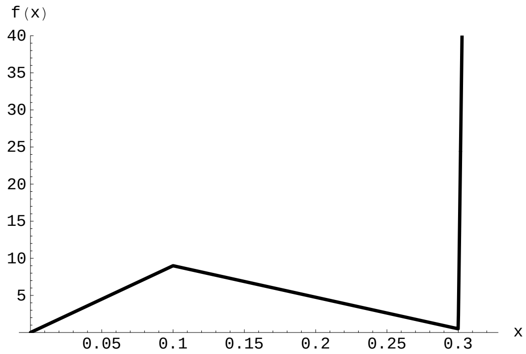

Figure 7: The graph of $y = f(x)$.

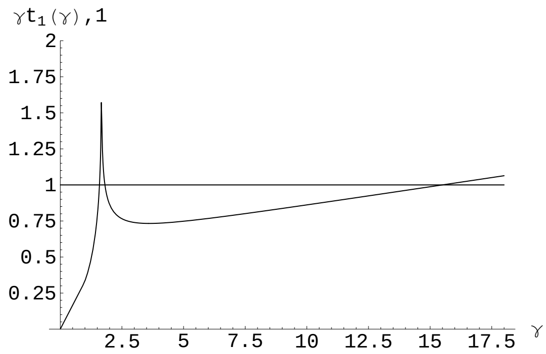

Figure 8: Graphs of $y = \gamma t_1(\gamma)$ and $y = 1$.

Function $\gamma t_1(\gamma)$ is nonmonotone under the line $y = 1$. Then a set of solutions of equation (27) may be decomposed in several components and behavior of branches of the spectrum may be relatively complicated.

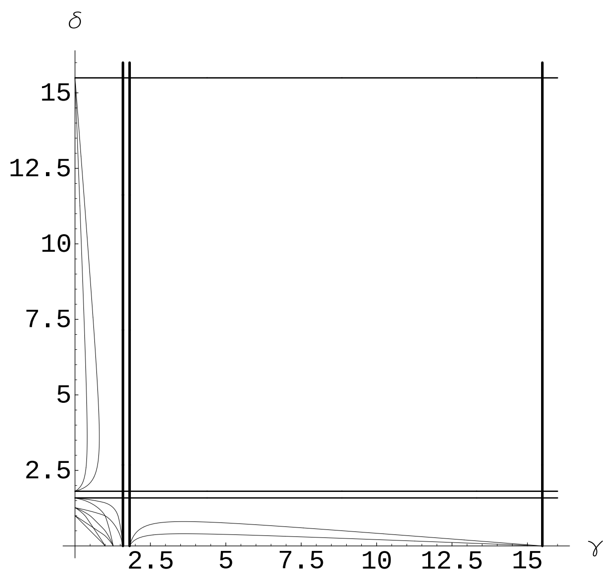

Figure 9: The branch $\mathcal{F}_0^+$ in the (γ, δ)-plane.

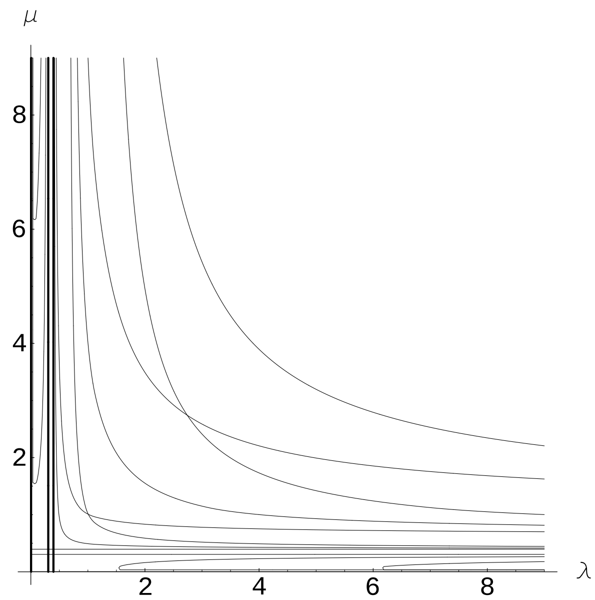

Figure 10: The branch F_0^+ in the (λ, μ)-plane.

The branch F_0^+ consists of three vertical lines which corresponds to three solutions of the equation $\frac{1}{\sqrt{\lambda}} t_1(\frac{1}{\sqrt{\lambda}}) = 1$.

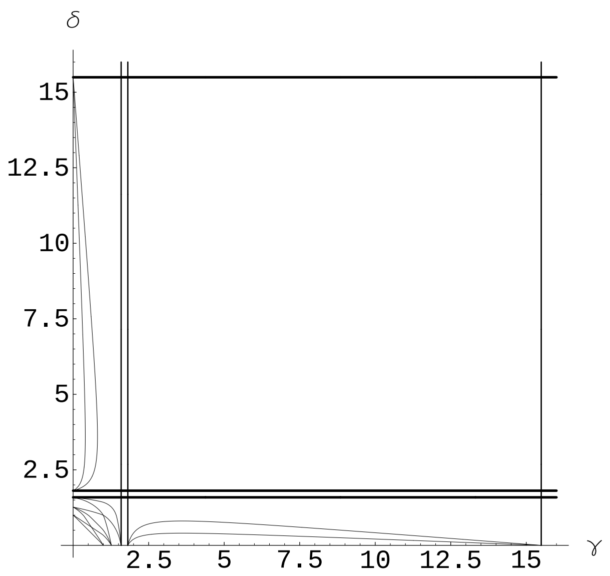

Figure 11: The branch $\mathcal{F}_0^-$ in the (γ, δ)-plane.

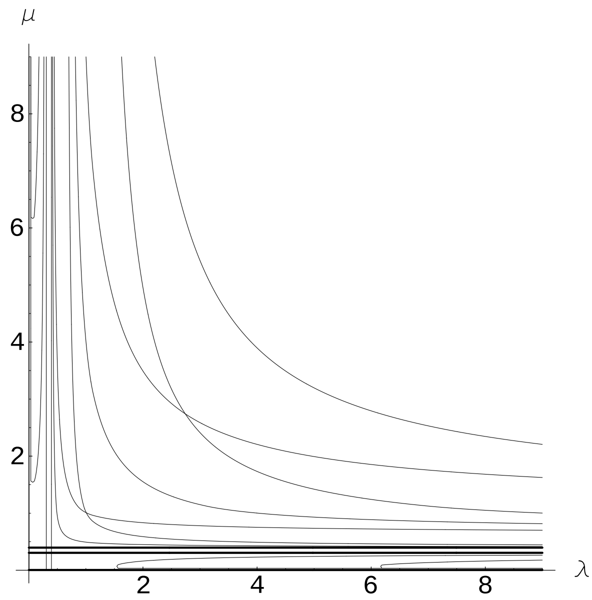

Figure 12: The branch F_0^- in the (λ, μ)-plane.

The same is true with respect to the branch F_0^-. It consists of horizontal lines which correspond to solutions of the equation $\frac{1}{\sqrt{\mu}}\tau_1(\frac{1}{\sqrt{\mu}}) = 1$.

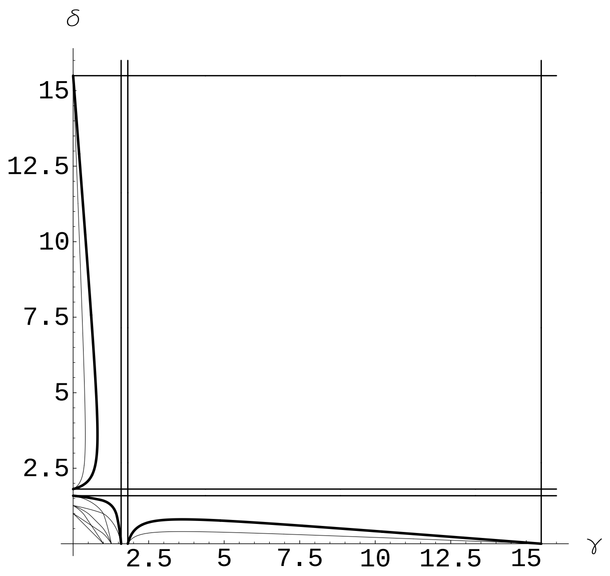

Figure 13: The branch $\mathcal{F}_1^+ = \mathcal{F}_1^-$ in the (γ, δ)-plane.

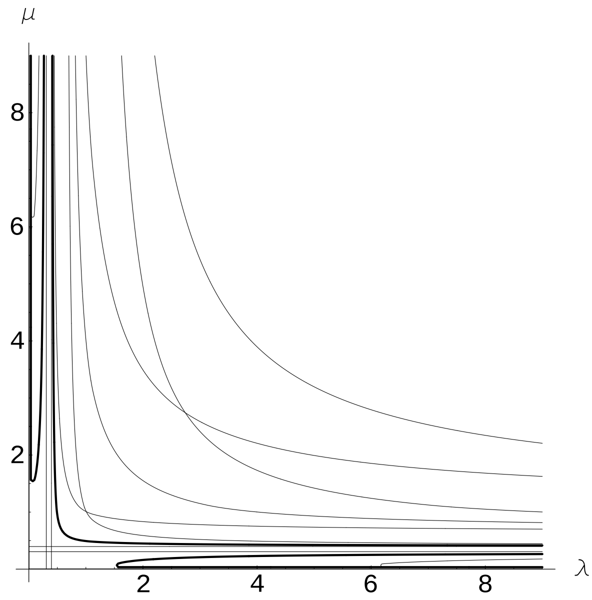

Figure 14: The branch $F_1^+ = F_1^-$ in the (λ, μ)-plane.

A set of solutions of equation (27) consists of exactly three components due to non-monotonicity of the functions $\frac{1}{\sqrt{\lambda}}t_1(\frac{1}{\sqrt{\lambda}})$ and $\frac{1}{\sqrt{\mu}}\tau_1(\frac{1}{\sqrt{\mu}})$. Properties of the branches $F_1^\pm$

depend on solutions of the equation

$$\frac{1}{\sqrt{\lambda}} t_1(\frac{1}{\sqrt{\lambda}}) + \frac{1}{\sqrt{\mu}} \tau_1(\frac{1}{\sqrt{\mu}}) = 1.$$

Respectively, properties of the branches $\mathcal{F}_1^{\pm}$ depend on solutions of the equation

$$\gamma t_1(\gamma) + \delta \tau_1(\delta) = 1.$$

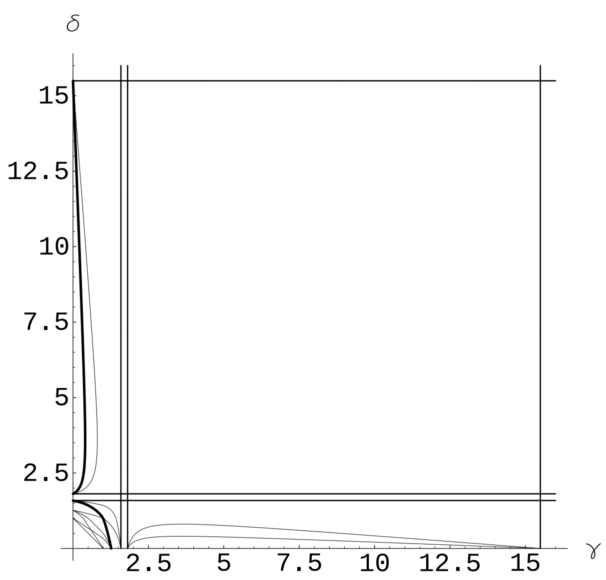

Figure 15: The branch $\mathcal{F}_2^+$ in the (γ, δ)-plane.

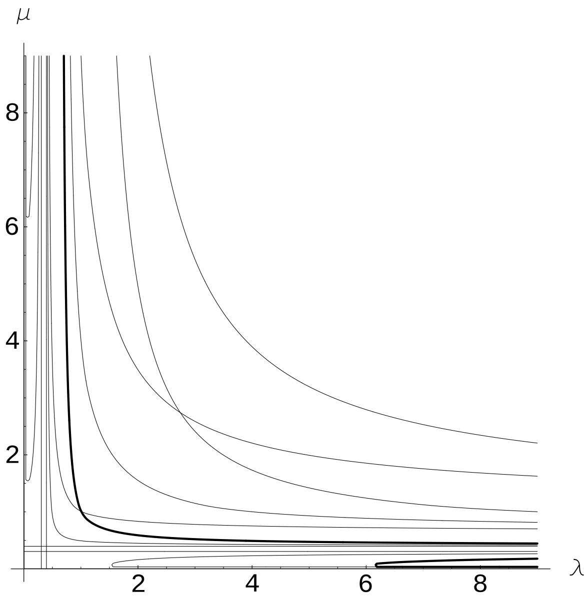

Figure 16: The branch F_2^+ in the (λ, μ)-plane.

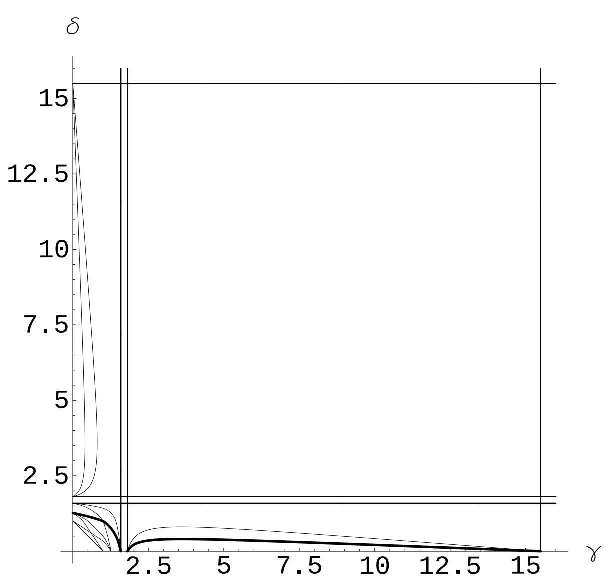

Figure 17: The branch $\mathcal{F}_2^-$ in the (γ, δ)-plane.

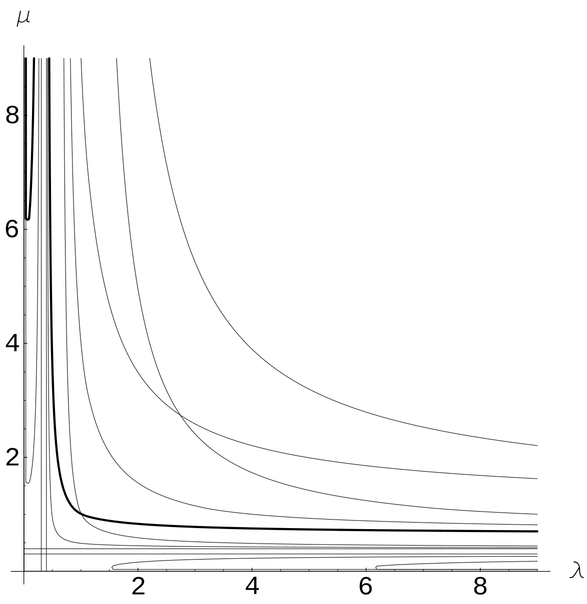

Figure 18: The branch F_2^- in the (λ, μ)-plane.

Branches $F_2^{\pm}$ look a little bit different since now their properties depend on a set of solutions of equations

$$2\frac{1}{\sqrt{\lambda}}t_1(\frac{1}{\sqrt{\lambda}}) + \frac{1}{\sqrt{\mu}}\tau_1(\frac{1}{\sqrt{\mu}}) = 1$$

and

$$\frac{1}{\sqrt{\lambda}}t_1(\frac{1}{\sqrt{\lambda}}) + 2\frac{1}{\sqrt{\mu}}\tau_1(\frac{1}{\sqrt{\mu}}) = 1.$$

Respectively, properties of the branches $\mathcal{F}_2^{\pm}$ depend on solutions of equations

$$2\gamma t_1(\gamma) + \delta\tau_1(\delta) = 1$$

and

$$\gamma t_1(\gamma) + 2\delta\tau_1(\delta) = 1.$$

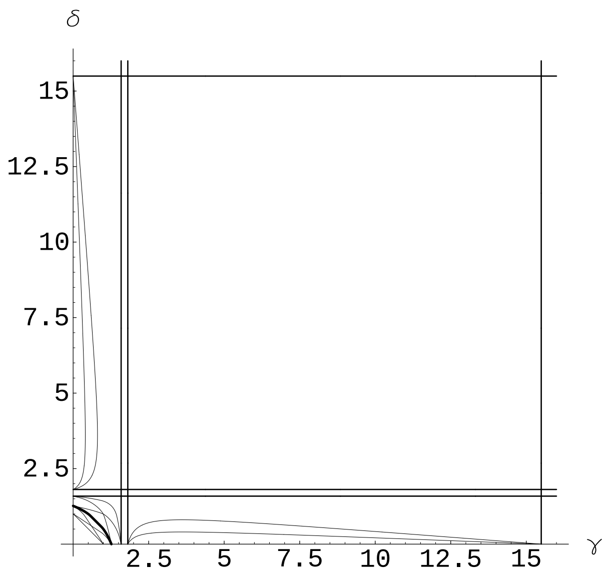

Figure 19: The branch $\mathcal{F}_3^+ = \mathcal{F}_3^-$ in the (γ, δ)-plane.

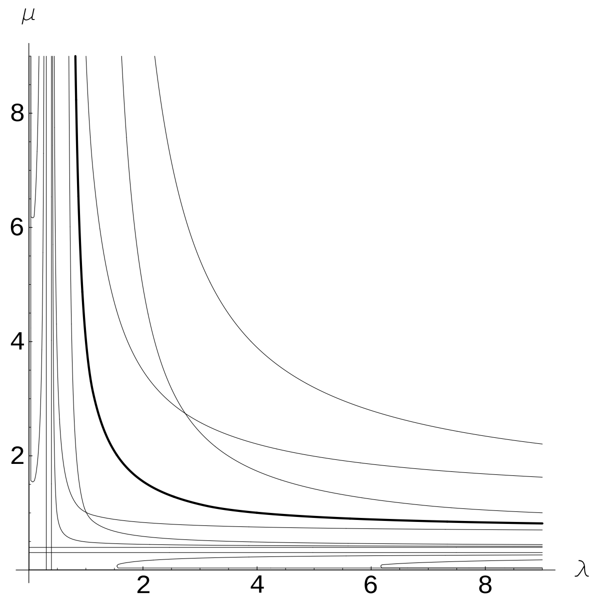

Figure 20: The branch $F_3^+ = F_3^-$ in the (λ, μ)-plane.

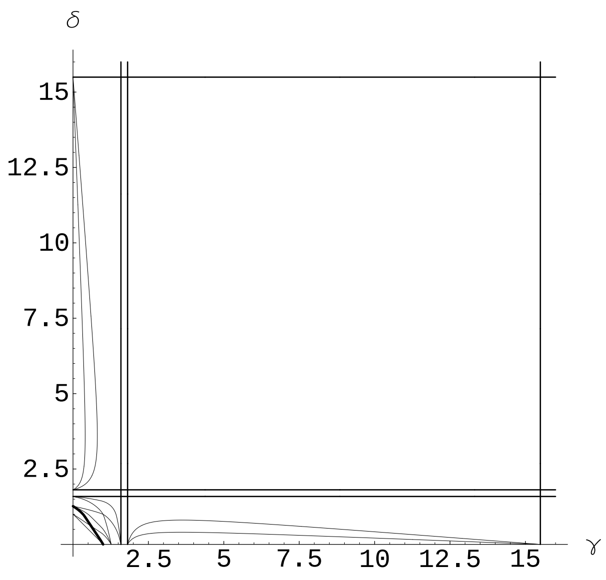

Figure 21: The branch $\mathcal{F}_4^+$ in the (γ, δ)-plane.

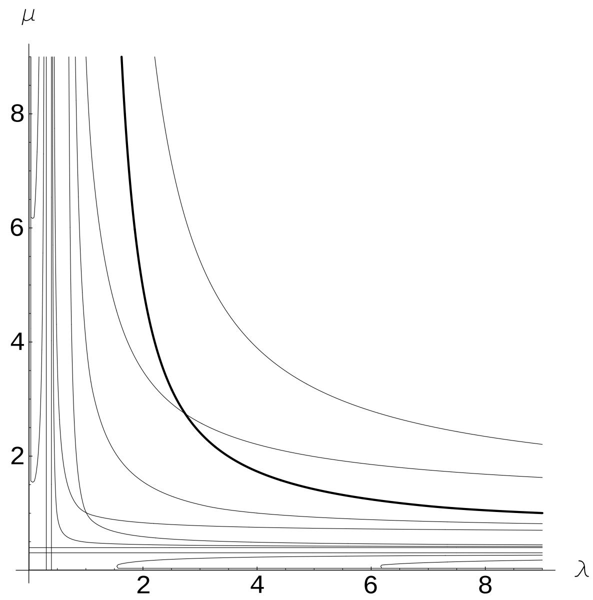

Figure 22: The branch F_4^+ in the (λ, μ)-plane.

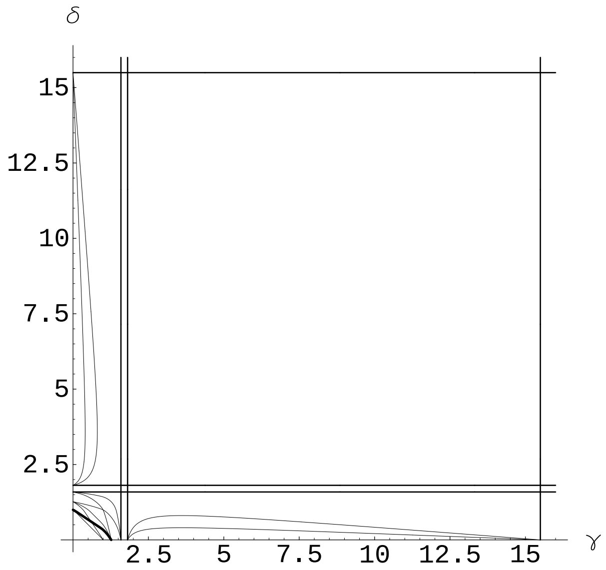

Figure 23: The branch $\mathcal{F}_4^-$ in the (γ, δ)-plane.

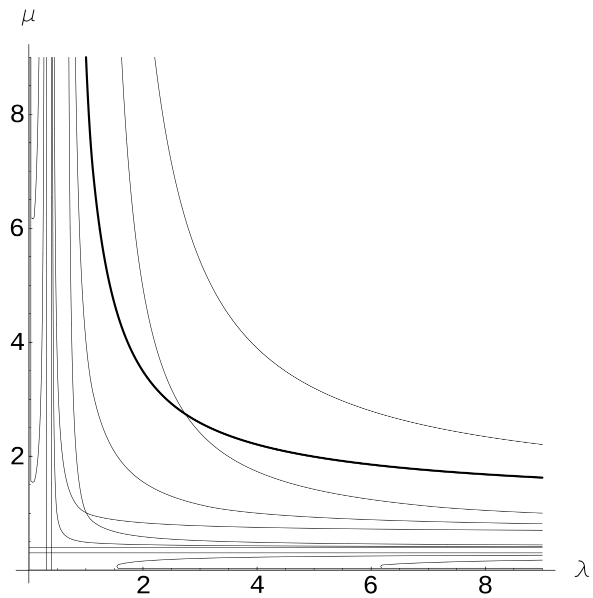

Figure 24: The branch F_4^- in the (λ, μ)-plane.

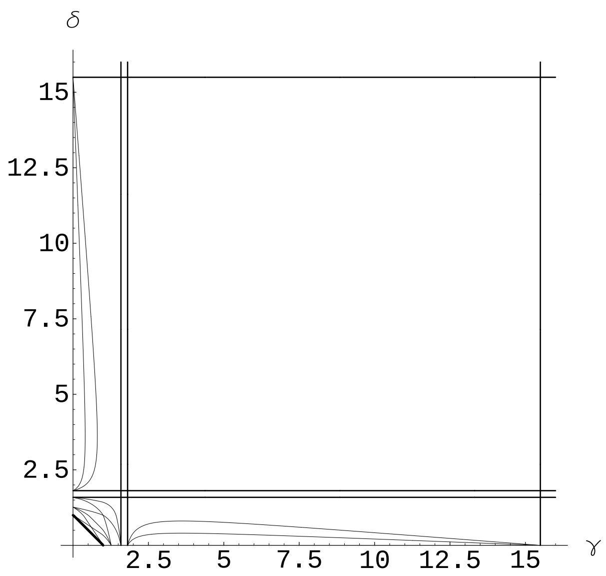

Figure 25: The branch $\mathcal{F}_5^+ = \mathcal{F}_5^-$ in the (γ, δ)-plane.

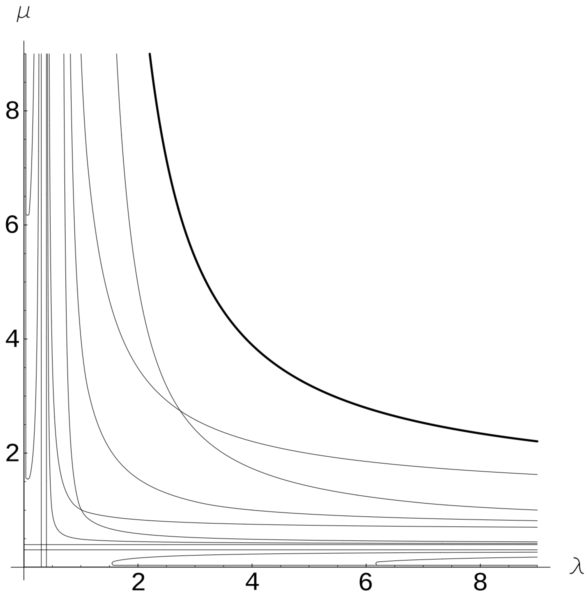

Figure 26: The branch $F_5^+ = F_5^-$ in the (λ, μ)-plane.

6 Discussion

- Fučik type spectra make sense for nonlinear functions f and g if the normalization condition $|x'(0)| = 1$ is imposed;
- The character of new Fučik type spectra essentially depend on properties of the first zero functions t_1 and τ_1;
- Some properties of the Fučik type spectra are better seen in a (γ, δ)-plane;
- Odd-numbered branches F_{2i-1}^+ and F_{2i-1}^- still coincide;
- Even-numbered branches F_{2i}^+ and F_{2i}^- generally differ;
- If functions $u(\gamma) = \gamma t_1(\gamma)$ and $v(\delta) = \delta \tau_1(\delta)$ are monotone then the respective Fučik type spectra are similar to the classical Fučik spectrum;
- If functions $u(\gamma) = \gamma t_1(\gamma)$ and/or $v(\delta) = \delta \tau_1(\delta)$ are not monotone then some branches of the respective Fučik type spectra may consist of multiple separate components, this feature is new comparing with classical spectrum;
- If functions $u(\gamma) = \gamma t_1(\gamma)$ and $v(\delta) = \delta \tau_1(\delta)$ both are monotone in some vicinity of zero then branches $F_k^{\pm}$ of a Fučik type spectrum are one-component for large enough values of k;
- If functions $u(\gamma)$ and/or $v(\delta)$ oscillate in some vicinity of zero then behavior of branches $F_k^{\pm}$ of a Fučik type spectrum may be complicated (one might think about construction of example);

- If $f(x) = g(-x)$ for $x > 0$ then branches (components) of a respective Fučik type spectrum are symmetric with respect to the bisectrix $\lambda = \mu$ or $\gamma = \delta$.

7 Appendix

Let the numbers a_i, b_j be such that

$$0 < a_1 < a_2 < a_3, \quad b_1 > b_2 > 0, \quad b_3 > b_2.$$

Consider a piece-wise linear function $f : [0; +\infty) \to [0; +\infty)$, which passes through the origin :

$$f(x) = \begin{cases} f_1(x), & 0 \le x \le a_1, \\ f_2(x), & a_1 \le x \le a_2, \\ f_3(x), & x \ge a_3, \end{cases}$$

$$f_1(x) = p_1 x + q_1, \quad f_2(x) = p_2 x + q_2, \quad f_3(x) = p_3 x + q_3,$$

$$f_1(0) = 0, \quad f_1(a_1) = f_2(a_1), \quad f_2(a_2) = f_3(a_2), \quad f_3(a_3) = b_3.$$

Notice that

$$\begin{aligned} p_1 &= \frac{b_1}{a_1}, & q_1 &= 0, \\ p_2 &= \frac{b_2 - b_1}{a_2 - a_1}, & q_2 &= \frac{b_1 a_2 - a_1 b_2}{a_2 - a_1}, \\ p_3 &= \frac{b_3 - b_2}{a_3 - a_2}, & q_3 &= \frac{b_2 a_3 - a_2 b_3}{a_3 - a_2}. \end{aligned}$$

Our intent is to investigate the first zero function $t_1(\alpha)$ of a solution $x(t; \alpha)$ of the Cauchy problem

$$x'' = -f(x), \tag{28}$$

$$x(0) = 0, \quad x'(0) = \alpha > 0. \tag{29}$$

This function is given by

$$t_1(\alpha) = \int_0^{x_\alpha} \frac{2ds}{\sqrt{\alpha^2 - 2F(s)}}, \tag{30}$$

where

$$F(x) = \int_0^x f(s)ds,$$

and x_α is the only positive zero of the equation

$$\alpha^2 - 2F(x) = 0.$$

The primitive F can be represented as

$$F(x) = \begin{cases} F_1(x) := \int\limits_0^x f_1(s)ds, & 0 \le x \le a_1, \\ F_2(x) := \int\limits_0^{a_1} f_1(s)ds + \int\limits_{a_1}^x f_2(s)ds, & a_1 \le x \le a_2, \\ F_3(x) := \int\limits_0^{a_1} f_1(s)ds + \int\limits_{a_1}^{a_2} f_2(s)ds + \int\limits_{a_2}^x f_3(s)ds, & x \ge a_3, \end{cases}$$

7.1 First step

Suppose that

$$0 \le \alpha^2 \le 2F(a_1) \quad \text{or} \quad 0 \le \alpha \le \sqrt{2F(a_1)}.$$

Since

$$F_1(x) = \frac{b_1}{2a_1} x^2,$$

then the only positive root of the equation

$$\alpha^2 - 2F_1(x) = 0 \quad \text{or} \quad \alpha^2 - \frac{b_1}{a_1} x^2 = 0$$

is given by

$$x_\alpha = \sqrt{\frac{a_1}{b_1}}\,\alpha.$$

Then

$$J_1(x) = \int\limits_0^x \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} = \int\limits_0^x \frac{2dt}{\sqrt{\alpha^2 - 2F_1(t)}} = \int\limits_0^x \frac{2dt}{\sqrt{\alpha^2 - \frac{b_1}{a_1} t^2}} =$$

$$= \int\limits_0^x \frac{2dt}{\sqrt{\alpha^2 - \left(\sqrt{\frac{b_1}{a_1}}\, t\right)^2}} = 2\sqrt{\frac{a_1}{b_1}} \int\limits_0^x \frac{d\left(\sqrt{\frac{b_1}{a_1}}\, t\right)}{\sqrt{\alpha^2 - \sqrt{\frac{b_1}{a_1}}\, t}} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\, t}{\alpha}\Bigg|_0^x =$$

$$= 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\, x}{\alpha}.$$

Therefore

$$t_1(\alpha) = \lim_{x \to x_\alpha} J_1(x) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\, x_\alpha}{\alpha} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\sqrt{\frac{a_1}{b_1}}\,\alpha}{\alpha} =$$

$$= 2\sqrt{\frac{a_1}{b_1}} \arcsin 1 = 2\sqrt{\frac{a_1}{b_1}}\,\frac{\pi}{2} = \pi\sqrt{\frac{a_1}{b_1}}$$

and

$$\boxed{\boldsymbol{t_1(\alpha) = \pi\sqrt{\frac{a_1}{b_1}}, \quad 0 \le \alpha \le \sqrt{2F(a_1)}}}$$

7.2 Second step

Suppose that

$$2F(a_1) \le \alpha^2 \le 2F(a_2) \quad \text{jeb} \quad \sqrt{2F(a_1)} \le \alpha \le \sqrt{2F(a_2)}.$$

Consider

$$J_2(x) = \int\limits_0^x \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} = \int\limits_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} + \int\limits_{a_1}^x \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} =$$
$$= \int\limits_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F_1(t)}} + \int\limits_{a_1}^x \frac{2dt}{\sqrt{\alpha^2 - 2F_2(t)}} = J_1(a_1) + J_{22}(x),$$

where

$$J_1(a_1) = \int\limits_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F_1(t)}} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\, a_1}{\alpha} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha},$$
$$J_{22}(x) = \int\limits_{a_1}^x \frac{2dt}{\sqrt{\alpha^2 - 2F_2(t)}}.$$

Notice that

$$G_2(t) := \alpha^2 - 2F_2(t) = r_2 t^2 + s_2 t + t_2(\alpha),$$

where

$$\begin{aligned} r_2 &= -\frac{b_2 - b_1}{a_2 - a_1} \\ &= -p_2 > 0, \\ s_2 &= \frac{2(a_2 b_1 - a_1 b_2)}{a_1 - a_2} \\ &= -2q_2 \\ &= -2f_2(0) < 0, \\ t_2 &= \alpha^2 - a_1 b_1 + \frac{a_1^2 b_1 - 2a_1 a_2 b_1 + a_1^2 b_2}{a_1 - a_2}. \end{aligned}$$

Equation

$$\alpha^2 - 2F(t) = 0$$

has the only positive root x_α *for any* $\alpha > 0$ *since the function* $F(t)$ *is monotonically increasing in the interval* $[0; +\infty)$, $F(0) = 0$.

If $2F(a_1) \le \alpha^2 \le 2F(a_2)$, then

$$\alpha^2 - 2F(t) = \alpha^2 - 2F_2(t) = \alpha^2 - 2F_2(t) = r_2 t^2 + s_2 t + t_2(\alpha) = G_2(t)$$

is quadratic polynomial. This means that x_α coincide to one of the quadratic polynomial's root, and

$$a_1 \le x_\alpha \le a_2.$$

The discriminant of the quadratic polynomial $G_2(t)$ is

$$D_2(\alpha) = s_2^2 - 4r_2t_2 \ge 0, \quad 2F(a_1) \le \alpha^2 \le 2F(a_2).$$

Notice that

$$D_2(\alpha) \ne 0.$$

► Suppose the contrary is true, that is, there exists $\alpha_0 \in \left[\sqrt{2F(a_1)}, \sqrt{2F(a_2)}\right]$ such that $D_2(\alpha_0) = 0$. Then

$$x_{\alpha_0} = \frac{-s_2}{2r_2} = \frac{2q_2}{-2p_2} = -\frac{q_2}{p_2}.$$

One has that

$$a_2 - x_{\alpha_0} = a_2 + \frac{q_2}{p_2} = \frac{p_2a_2 + q_2}{p_2} = \frac{f_2(a_2)}{p_2} < 0,$$

since $f_2(a_2) > 0$, bet $p_2 < 0$. Thus $x_{\alpha_0} > a_2$, and this is in contradiction with $x_{\alpha_0} \le a_2$.◄

Moreover,

$$x_\alpha = \frac{-s_2 - \sqrt{D_2(\alpha)}}{2r_2}.$$

Notice that if $a > 0$ un $D = b^2 - 4ac \ge 0$, then

$$ax^2 + bx + c = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \frac{D}{4a}.$$

One finds taking this into account that

$$\int \frac{2dt}{\sqrt{ax^2+bx+c}} = \int \frac{2dt}{\sqrt{\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \frac{D}{4a}}} = \frac{2}{\sqrt{a}} \int \frac{d\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)}{\sqrt{\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \frac{D}{4a}}} =$$

$$= \frac{2}{\sqrt{a}} \ln\left|\sqrt{a}x + \frac{b}{2\sqrt{a}} + \sqrt{ax^2+bx+c}\right| + C = \frac{2}{\sqrt{a}} \ln\left|2ax + b + 2\sqrt{a}\sqrt{ax^2+bx+c}\right| + C_1.$$

Hence

$$J_{22}(x) = \int\limits_{a_1}^{x} \frac{2dt}{\sqrt{\alpha^2 - 2F_2(t)}} = \int\limits_{a_1}^{x} \frac{2dt}{\sqrt{r_2t^2 + s_2t + t_2(\alpha)}} =$$

$$= \frac{2}{\sqrt{r_2}} \ln\left|2r_2t + s_2 + 2\sqrt{r_2}\sqrt{r_2t^2 + s_2t + t_2(\alpha)}\right|\Bigg|_{a_1}^{x} =$$

$$= \frac{2}{\sqrt{r_2}}\left[\ln\left|2r_2x + s_2 + 2\sqrt{r_2}\sqrt{r_2x^2 + s_2x + t_2(\alpha)}\right| - \ln\left|2r_2a_1 + s_2 + 2\sqrt{r_2}\sqrt{r_2a_1^2 + s_2a_1 + t_2(\alpha)}\right|\right] =$$

$$= \frac{2}{\sqrt{r_2}}\left[\ln\left|2r_2x + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(x)}\right| - \ln\left|2r_2a_1 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_1)}\right|\right].$$

It is of interest to note that

$$r_2a_1^2+s_2a_1+t_2(\alpha)=\alpha^2-2F_2(a_1)=\alpha^2-2F_1(a_1)=\alpha^2-a_1b_1.$$

Therefore

$$t_1(\alpha)=\lim_{x\to x_\alpha}J_2(x)=J_1(a_1)+\lim_{x\to x_\alpha}J_{22}(x)=$$

$$=\left|\ \alpha^2-2F_2(x_\alpha)=r_2x_\alpha^2+s_2x_\alpha+t_2(\alpha)=0\ \right|=$$

$$=J_1(a_1)+\frac{2}{\sqrt{r_2}}\left[\ln|2r_2x_\alpha+s_2|-\ln\left|2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right|\right]=$$

$$=\left|\ 2r_2x_\alpha+s_2=2r_2\tfrac{-s_2-\sqrt{D_2(\alpha)}}{2r_2}+s_2=-\sqrt{D_2(\alpha)}\ \right|=$$

$$=J_1(a_1)+\frac{2}{\sqrt{r_2}}\left[\ln\left|-\sqrt{D_2(\alpha)}\right|-\ln\left|2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right|\right]=$$

$$=J_1(a_1)+\frac{2}{\sqrt{r_2}}\left[\ln\sqrt{D_2(\alpha)}-\ln\left|2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right|\right]=$$

$$=J_1(a_1)+\frac{1}{\sqrt{r_2}}\left[\ln D_2(\alpha)-2\ln\left|2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right|\right]=$$

$$=J_1(a_1)+\frac{1}{\sqrt{r_2}}\left[\ln D_2(\alpha)-\ln\left(2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right)^2\right].$$

Notice that

$$2r_2a_1+s_2=-2(p_2a_1+q_2)=-2f_2(a_1)=-2b_1.$$

Therefore

$$\boldsymbol{\sqrt{2F(a_1)}\le\alpha\le\sqrt{2F(a_2)},}$$

$$\boldsymbol{t_1(\alpha)=2\sqrt{\frac{a_1}{b_1}}\arcsin\frac{\sqrt{a_1b_1}}{\alpha}+\frac{1}{\sqrt{r_2}}\ln\frac{D_2(\alpha)}{\left(-2b_1+2\sqrt{r_2}\sqrt{\alpha^2-a_1b_1}\right)^2}.}$$

This can be written also as

$$\boldsymbol{\sqrt{2F(a_1)}\le\alpha\le\sqrt{2F(a_2)},}$$

$$\boldsymbol{t_1(\alpha)=2\sqrt{\frac{a_1}{b_1}}\arcsin\frac{\sqrt{a_1b_1}}{\alpha}+}$$

$$\boldsymbol{+\frac{2}{\sqrt{r_2}}\left[\ln\left|2r_2x_\alpha+s_2+2\sqrt{r_2}\sqrt{\alpha^2-2F_2(x_\alpha)}\right|-\ln\left|2r_2a_1+s_2+2\sqrt{r_2}\sqrt{\alpha^2-2F_2(a_1)}\right|\right].}$$

7.3 Third step

Suppose that

$$\alpha^2 \geq 2F(a_2) \quad \text{or} \quad \alpha \geq \sqrt{2F(a_2)}.$$

Consider

$$J_3(x) = \int_0^x \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} =$$

$$\int_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} + \int_{a_1}^{a_2} \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} + \int_{a_2}^{x} \frac{2dt}{\sqrt{\alpha^2 - 2F(t)}} =$$

$$\int_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F_1(t)}} + \int_{a_1}^{a_2} \frac{2dt}{\sqrt{\alpha^2 - 2F_2(t)}} + \int_{a_2}^{x} \frac{2dt}{\sqrt{\alpha^2 - 2F_3(t)}} =$$

$$= J_1(a_1) + J_{22}(a_2) + J_{33}(x),$$

where

$$J_1(a_1) = \int_0^{a_1} \frac{2dt}{\sqrt{\alpha^2 - 2F_1(t)}} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{\frac{b_1}{a_1}}\, a_1}{\alpha} = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha},$$

$$J_{22}(a_2) = \int_{a_1}^{a_2} \frac{2dt}{\sqrt{\alpha^2 - 2F_2(t)}} =$$

$$= \frac{2}{\sqrt{r_2}} \left[\ln \left| 2r_2 a_2 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_2)} \right| - \ln \left| 2r_2 a_1 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_1)} \right| \right],$$

$$J_{33}(x) = \int_{a_2}^{x} \frac{2dt}{\sqrt{\alpha^2 - 2F_3(t)}}.$$

Notice that

$$2r_2 a_2 + s_2 = -2(p_2 a_2 + q_2) = -2f_2(a_2) < 0.$$

Consider

$$G_3(t) := \alpha^2 - 2F_3(t) = r_3 t^2 + s_3 t + t_3(\alpha),$$

where

$$\begin{aligned} r_3 &= -\frac{b_3 - b_2}{a_3 - a_2} \\ &= -p_3 < 0, \\ s_3 &= \frac{2(a_3 b_2 - a_2 b_3)}{a_2 - a_3} \\ &= -2q_3 \\ &= -2f_3(0), \\ t_3 &= \alpha^2 - a_2 b_1 + a_1 b_2 - a_2 b_2 + \frac{a_2^2 b_2 - 2a_2 a_3 b_2 + a_2^2 b_3}{a_2 - a_3} = \alpha^2 + const. \end{aligned}$$

Since $F(t)$ is strictly monotonic in $[0;+\infty)$, $F(0)=0$, there exists a unique solution x_α of the equation

$$\alpha^2 - 2F(t) = 0$$

for any $\alpha > 0$.

If $\alpha^2 \geq 2F(a_2)$, then

$$\alpha^2 - 2F(t) = \alpha^2 - 2F_3(t) = \alpha^2 - 2F_3(t) = r_3t^2 + s_3t + t_3(\alpha) = G_3(t)$$

is a quadratic polynomial. This means that x_α coincide with one of the roots of the quadratic polynomial and

$$x_\alpha \geq a_2.$$

The discriminant $G_3(t)$ of the quadratic polynomial is

$$D_3(\alpha) = s_3^2 - 4r_3t_3 \geq 0, \quad \alpha^2 \geq 2F(a_2).$$

Notice that

$$D_3(\alpha) \neq 0.$$

► Suppose the contrary is true. Then there exists $\alpha_0 \in \left[\sqrt{2F(a_3)}, +\infty\right]$ such that $D_3(\alpha_0) = 0$. Then

$$x_{\alpha_0} = \frac{-s_3}{2r_3} = \frac{-(2q_3)}{-2p_2} = -\frac{q_3}{p_3}.$$

One finds that

$$x_{\alpha_0} - a_2 = -\frac{q_3}{p_3} - a_2 = -\frac{p_3a_2 + q_3}{p_3} = -\frac{f_3(a_2)}{p_3} < 0,$$

since $f_2(a_3) > 0$, $p_3 > 0$. Therefore $x_{\alpha_0} < a_2$, and this is in contradiction with $x_{\alpha_0} \geq a_2$.◄

Moreover

$$x_\alpha = \frac{-s_3 - \sqrt{D_3(\alpha)}}{2r_3}.$$

Notice that if $a < 0$ and $D = b^2 - 4ac \geq 0$, then

$$\begin{aligned}
ax^2 + bx + c &= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} \\
&= -|a|\left(x + \frac{b}{2a}\right)^2 - \frac{D}{-4|a|} \\
&= \frac{D}{4|a|} - |a|\left(x + \frac{b}{2a}\right)^2 \\
&= |a|\left[\frac{D}{4|a|^2} - \left(x + \frac{b}{2a}\right)^2\right] \\
&= |a|\left[\left(\frac{\sqrt{D}}{2|a|}\right)^2 - \left(x + \frac{b}{2a}\right)^2\right].
\end{aligned}$$

It follows from the above relation that

$$\int \frac{2dx}{\sqrt{ax^2+bx+c}} = \int \frac{2dx}{\sqrt{|a|\left[\left(\frac{\sqrt{D}}{2|a|}\right)^2 - \left(x+\frac{b}{2a}\right)^2\right]}} = \frac{2}{\sqrt{|a|}} \int \frac{d\left(x+\frac{b}{2a}\right)}{\sqrt{\left(\frac{\sqrt{D}}{2|a|}\right)^2 - \left(x+\frac{b}{2a}\right)^2}} =$$

$$= \frac{2}{\sqrt{|a|}} \arcsin \frac{x+\frac{b}{2a}}{\frac{\sqrt{D}}{2|a|}} = \frac{2}{\sqrt{-a}} \arcsin \frac{x+\frac{b}{2a}}{-\frac{\sqrt{D}}{2a}} = \frac{2}{\sqrt{-a}} \arcsin \frac{2ax+b}{-\sqrt{D}}.$$

Then

$$J_{33}(x) = \int\limits_{a_2}^{x} \frac{2dt}{\sqrt{\alpha^2 - 2F_3(t)}} = \int\limits_{a_3}^{x} \frac{2dt}{\sqrt{r_3 t^2 + s_3 t + t_3(\alpha)}} = \frac{2}{\sqrt{-r_3}} \arcsin \frac{2r_3 t + s_3}{-\sqrt{D_3(\alpha)}} \Bigg|_{a_2}^{x} =$$

$$\frac{2}{\sqrt{-r_3}} \left[\arcsin \frac{2r_3 x + s_3}{-\sqrt{D_3(\alpha)}} - \arcsin \frac{2r_3 a_3 + s_3}{-\sqrt{D_3(\alpha)}}\right].$$

Therefore

$$t_1(\alpha) = \lim_{x\to x_\alpha} J_3(x) = J_1(a_1) + J_{22}(a_2) + \lim_{x\to x_\alpha} J_{33}(x) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} +$$

$$+ \frac{2}{\sqrt{r_2}} \left[\ln\left|2r_2 a_2 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_2)}\right| - \ln\left|2r_2 a_1 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_1)}\right|\right] +$$

$$+ \frac{2}{\sqrt{-r_3}} \left[\arcsin \frac{2r_3 x_\alpha + s_3}{-\sqrt{D_3(\alpha)}} - \arcsin \frac{2r_3 a_2 + s_3}{-\sqrt{D_3(\alpha)}}\right] =$$

$$= \left| \frac{2r_3 x_\alpha + s_3}{-\sqrt{D_3(\alpha)}} = \frac{2r_3 \frac{-s_3 - \sqrt{D_3(\alpha)}}{2r_3} + s_3}{-\sqrt{D_3(\alpha)}} = 1 \right| =$$

$$= 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \frac{\pi}{\sqrt{-r_3}} - \frac{2}{\sqrt{-r_3}} \arcsin \frac{2r_3 a_2 + s_3}{-\sqrt{D_3(\alpha)}} +$$

$$+ \frac{2}{\sqrt{r_2}} \left[\ln\left|2r_2 a_2 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_2)}\right| - \ln\left|2r_2 a_1 + s_2 + 2\sqrt{r_2}\sqrt{\alpha^2 - 2F_2(a_1)}\right|\right].$$

Notice that

$$r_2 = -p_2 = -\frac{b_2 - b_1}{a_2 - a_1} = \frac{b_1 - b_2}{a_2 - a_1};$$

$$-r_3 = -(-p_3) = p_3 = \frac{b_3 - b_2}{a_3 - a_2};$$

$$2r_2 a_1 + s_2 = 2(-p_2)a_1 + (-2q_2) = -2(p_2 a_1 + q_2) = -2f_2(a_1) = -2b_1;$$

$$2r_2 a_2 + s_2 = 2(-p_2)a_2 + (-2q_2) = -2(p_2 a_2 + q_2) = -2f_2(a_2) = -2b_2;$$

$$2r_3 a_2 + s_3 = 2(-p_3)a_2 + (-2q_3) = -2(p_3 a_2 + q_3) = -2f_3(a_2) = -2b_2;$$

Thus

$$\boldsymbol{t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}}\arcsin\frac{\sqrt{a_1b_1}}{\alpha} + \sqrt{\frac{a_3-a_2}{b_3-b_2}}\left[\pi - 2\arcsin\frac{2b_2}{\sqrt{D_3(\alpha)}}\right] +}$$

$$\boldsymbol{+\,2\sqrt{\frac{a_2-a_1}{b_1-b_2}}\ln\left|\frac{-b_2+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_2)}}{-b_1+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_1)}}\right|, \quad \alpha \geq \sqrt{2F_2(a_2)}.}$$

If $\alpha \to +\infty$, then

$$2\sqrt{\frac{a_1}{b_1}}\arcsin\frac{\sqrt{a_1b_1}}{\alpha} \to 0.$$

Since

$$D_3(\alpha) = s_3^2 - 4r_3t_3(\alpha) = s_3^2 - 4r_3(\alpha^2 + const) = -4r_3\alpha^2 + const,$$

where $r_3 < 0$, then $D_3(\alpha) \to +\infty$, if $\alpha \to +\infty$. Therefore

$$2\arcsin\frac{2b_2}{\sqrt{D_3(\alpha)}} \xrightarrow[\alpha\to+\infty]{} 0.$$

Moreover, since

$$\frac{-b_2+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_2)}}{-b_1+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_1)}} \xrightarrow[\alpha\to+\infty]{} 1,$$

then

$$\ln\left|\frac{b_2+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_2)}}{b_1+\sqrt{\frac{b_1-b_2}{a_2-a_1}}\sqrt{\alpha^2-2F_2(a_1)}}\right| \xrightarrow[\alpha\to+\infty]{} 0.$$

Therefore

$$\boldsymbol{t_1(\alpha) \xrightarrow[\alpha\to+\infty]{} \sqrt{\frac{a_3-a_2}{b_3-b_2}}\pi.}$$

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А. Грицанс, Ф. Садырбаев. Нелинейные спектры двупараметрических задач на собственные значения.

Аннотация. Рассматриваются задачи на собственные значения вида $x'' = -\lambda f(x) + \mu g(x)$, (i), $x(0) = 0$, $x(1) = 0$, (ii). Ищутся значения (λ, μ) такие, что задача $(i), (ii)$ имеет нетривиальное решение. Эта задача обобщает известную задачу Фучика для кусочно-линейных уравнений. С целью показать, что нелинейные спектры Фучика могут существенно отличаться от классических, мы рассматриваем функции $f(x)$ и $g(x)$ кусочно-линейными, что дает возможность получить явные формулы для функций первых нулей t_1 и τ_1. Отсюда получаются явные выражения для соответствующих спектров Фучика.

УДК 517.927

A. Gricāns, F. Sadirbajevs. Divu parametru īpašvērtību problēmu nelineārie spektri.

Anotācija. Tika apskatītas īpašvērtību problēmas formā $x'' = -\lambda f(x) + \mu g(x), \quad (i),$ $x(0) = 0,\ x(1) = 0, \quad (ii)$. Meklējam (λ, μ) tādus, ka problēmai $(i), (ii)$ ir netriviālie atrisinājumi. Šis uzdevums vispārina slavenu Fučika problēmu kad $f(x) = x^+$ un $g(x) = x^-$. Mēs iegūstam tiešas formulas Fučika spektriem gadījumā, kad funkcijas $f(x)$ un $g(x)$ katra ir gabaliem lineāras funkcijas. Iegūtie spektri būtiski atšķiras no klasiskiem.

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Received 02.05.2007