On solutions of 6-th order linear differential equations [1]

Tatjana Garbuza

Daugavpils University, Daugavpils, Latvia, garbuza@inbox.lv

Summary. We consider some important properties of solutions of linear differential equations of 6-th order.

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1 Introduction

In this paper we consider linear differential equations of the form

$$x^{(6)} = -p(t)x, \tag{1}$$

$$x^{(6)} = p(t)x, \tag{2}$$

where $p(t)$ is a positive valued continuous function. We are interested in oscillatory behavior of solutions, conjugate points and related things.

First we study the representatives of these two classes of equations, namely, the equations

$$x^{(6)} = -x \tag{3}$$

and

$$x^{(6)} = x, \tag{4}$$

which can be solved explicitly.

In the second part of the work we formulate some assertions about solutions of equations (1) and (2).

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2 Preliminaries and analogies

The most studied second order equation

$$x'' = p(t)x \tag{5}$$

with $p \in C(R)$ is oscillatory, if its solutions have zeros. If p is positive than no solutions with more than one zero exist. The example is $x'' = x$. If p is negative valued than oscillatory solutions can exist. The example is $x'' = -x$. The trigonometric functions $\sin t$ and $\cos t$ and linear combinations are solutions.

For solutions of higher order equations, that is,

$$x^{(n)} = p(t)x, \quad n > 2, \tag{6}$$

oscillation can be defined in terms of conjugate points.

Definition 2.1 *By m-th conjugate point of a is called the minimal value of a_{m+n-1}, where a_{m+n-1} is a $(m+n-1)$-th zero of solutions of* (6) *which have a zero at $t = a$, counting multiplicities. [2]*

Since conjugate points are infinums of some zeros, and therefore possess some extremal property, the related solutions are called *extremal solutions* .

The idea of investigation of oscillatory properties of such equations in terms of conjugate points goes back to W. Leighton and Z. Nehari [3], who investigated linear equations of the type

$$x^{(4)} = p(t)x. \tag{7}$$

In general case much is known about the first conjugate point η_1 and little about n-th conjugate point.

3 Equations (3) and (4)

3.1 Equation (3)

Consider equation (3) in a greater generality

$$x^{(6)} = -k^6 x. \tag{8}$$

The general solution is

$$x(t) = C_1 \exp\left(\frac{\sqrt{3}}{2}kt\right)\cos\frac{kt}{2} + C_2 \exp\left(-\frac{\sqrt{3}}{2}kt\right)\cos\frac{kt}{2} + C_3\cos kt + C_4 \sin kt + \\ + C_5 \exp\left(\frac{\sqrt{3}}{2}kt\right)\sin\frac{kt}{2} + C_6 \exp\left(-\frac{\sqrt{3}}{2}kt\right)\sin\frac{kt}{2}. \tag{9}$$

Computations show that there exist solutions of the (8) with exactly two triple zeros at $t = 0$ and $t = \eta_{m+1}$ such that there are exactly m simple zeros in $(0, \eta_{m+1})$. We don't know either these points are conjugate points to $t = 0$ in the sense of Definition 2.1 or not. Denote these solutions as $(3,3)$-solutions.

We show computations for the case of $k = 1$ and obtain the problem

$$\begin{cases} x^{(6)} = -x, \\ x(0) = x'(0) = x''(0) = 0, \\ x'''(0) = \alpha, \\ x^{IV}(0) = \beta, \\ x^{V}(0) = \gamma. \end{cases} \tag{10}$$

The solution of this problem is

$$x(t,\alpha,\beta,\gamma) = \frac{2\alpha+\sqrt{3}\beta+\gamma}{6}\exp(\frac{\sqrt{3}}{2}t)\sin\frac{t}{2} - \frac{\beta+\sqrt{3}\gamma}{6}\exp(\frac{\sqrt{3}}{2}t)\cos\frac{t}{2}+ \\ + \frac{2\alpha-\sqrt{3}\beta+\gamma}{6}\exp(-\frac{\sqrt{3}}{2}t)\sin\frac{t}{2} - \frac{\beta-\sqrt{3}\gamma}{6}\exp(-\frac{\sqrt{3}}{2}t)\cos\frac{t}{2}+ \\ + \frac{\beta}{3}\cos t + \frac{\gamma-\alpha}{3}\sin t.$$

The linear homogeneous system with respect to unknown α, β, γ

$$\begin{cases} x(\eta,\alpha,\beta,\gamma) = 0, \\ x'(\eta,\alpha,\beta,\gamma) = 0, \\ x''(\eta,\alpha,\beta,\gamma) = 0 \end{cases}$$

has a nontrivial solution if and only if the determinant of system is zero. After computations we have

$$16\cosh\frac{\sqrt{3}}{2}\eta\sin\frac{\eta}{2} - 2(4+\cosh\sqrt{3}\eta)\sin\eta + \sin 2\eta = 0,$$

or

$$\sin\frac{\eta}{2}\left[16\cosh\frac{\sqrt{3}}{2}\eta - 4(4+\cosh\sqrt{3}\eta)\cos\frac{\eta}{2} + 4\cos\eta\cos\frac{\eta}{2}\right] = 0.$$

This equation has infinitely many zeros, since all values $2\pi n$, $n \in N$, are zeros. The first ten values of η_i and (α,β,γ) which relate to η_i, are given in the Table 1.

Table 1. The initial values of solutions of (10) which have triple zeros at $t = 0$ and $t = \eta_i$.

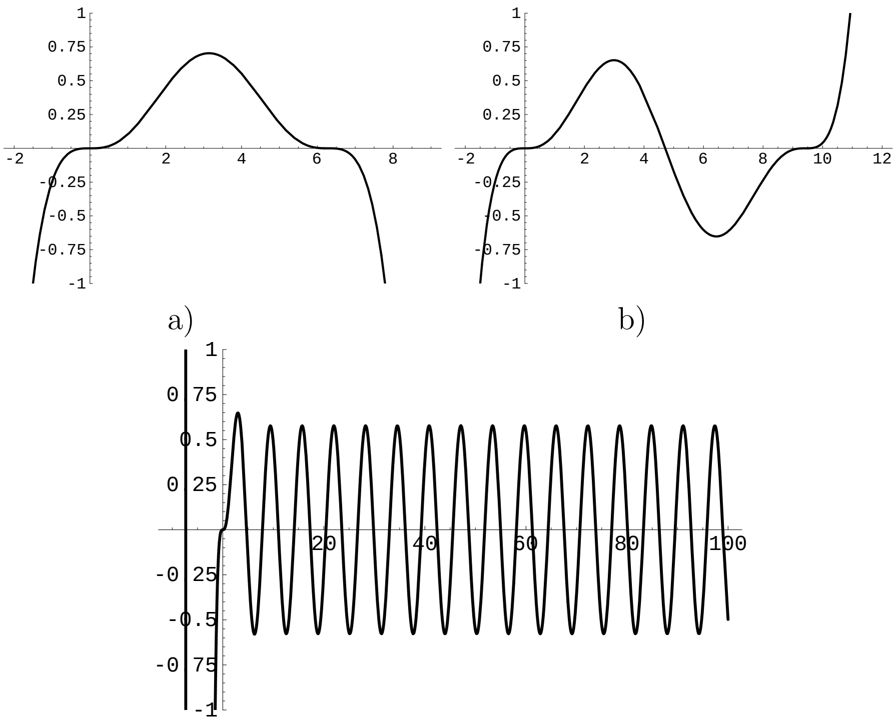
η_i	η_i value	$x'''(0) = \alpha$	$x^{IV}(0) = \beta$	$x^{V}(0) = \gamma$
η_1	2π	1	-1.717104168	1
η_2	9.427055570888907	1.001974415352	-1.733759858	1
η_3	4π	1	-1.73211585	1
η_4	15.707953378529623	0.9999914355482	-1.732043390	1
η_5	6π	1	-1.732050525	1
η_6	21.991148617983196	1.0000000371132	-1.732050839	1
η_7	8π	1	-1.732050808	1
η_8	28.27433388212243	0.9999999998391	-1.732050807	1
η_9	10π	1	-1.732050807	1
η_{10}	34.55751918948853	1.0000000000006	-1.732050807	1

All solutions with such initial conditions (α, β, γ) tend to a solution $x^*(t)$

$$x^*(t) = -\frac{1}{\sqrt{3}} \cos t + \frac{1}{3} \exp(-\frac{\sqrt{3}}{2} t) \left(\sqrt{3} \cos \frac{t}{2} + 3 \sin \frac{t}{2} \right).$$

$x^*(t)$ satisfy the initial conditions $\alpha = 1$, $\beta = -\sqrt{3}$, $\gamma = 1$. (See 3.1)

The same is true for the equation $x^{(6)} = -k^6 x$. This can be shown by the independent variable change $\tau = kt$.



c)

Figure 3.1: a) The solution $x(t)$ of (10) which relates to η_1; b) The solution $x(t)$ of (10) which relates to η_2; c) The solution of (10) $x^*(t)$.

Considerations show that solutions $x_i(t)$ with triple zeros at $t = 0$ and $t = \eta_i$ have the following structure: any solution x_i has exactly $i - 1$ simple zeros in the interval $(0, \eta_i)$; the initial values $(x'''(0), x^{(4)}(0), x^{(5)}(0))$ tend to those of a specific solution $x^*(t)$ (see Fig. 3.1).

3.2 Equation (4)

Consider equation (4) in a greater generality

$$x^{(6)} = k^6 x. \tag{11}$$

The general solution is

$$x(t) = C_1 \exp \frac{kt}{2} \cos \frac{\sqrt{3}}{2}kt + C_2 \exp\left(-\frac{kt}{2}\right) \cos \frac{\sqrt{3}}{2}kt + C_3 \exp kt + C_4 \exp(-kt) + \\ + C_5 \exp \frac{kt}{2} \sin \frac{\sqrt{3}}{2}kt + C_6 \exp\left(-\frac{kt}{2}\right) \sin \frac{\sqrt{3}}{2}kt. \quad (12)$$

Consider solutions of (11), which satisfy the initial conditions

$$x^{(i)}(0) = 0, \quad i = 0, 1, 2, 3, \quad (13)$$

$$x^{(4)}(0) = \alpha, \quad x^{(5)}(0) = \beta. \quad (14)$$

We provide some simple results on the linear equation (11).

Computations show that there exist solutions of the initial value problem (11), (13), (14), which have double zero for $t > 0$. We don't know either these points are conjugate points to $t = 0$ in the sense of Definition 2.1 or not. Denote these points $\eta_i(k)$, and corresponding solutions as $(4, 2)$-solutions.

We show computations for the case of $k = 1$ and obtain the problem

$$\begin{cases} x^{(6)} = -x, \\ x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ x^{IV}(0) = \beta, \\ x^{V}(0) = \gamma. \end{cases} \quad (15)$$

The solution of this problem is

$$x(t, \beta, \gamma) = \frac{\beta}{3}\left(\cosh t - \cosh \frac{t}{2} \cos \frac{\sqrt{3}}{2}t - \sqrt{3} \sinh \frac{t}{2} \sin \frac{\sqrt{3}}{2}t\right) + \\ + \frac{\gamma}{3}\left(\sinh t + \sinh \frac{t}{2} \cos \frac{\sqrt{3}}{2}t - \sqrt{3} \cosh \frac{t}{2} \sin \frac{\sqrt{3}}{2}t\right). \quad (16)$$

The linear homogeneous system with respect to unknown β, γ

$$\begin{cases} x(\eta, \beta, \gamma) = 0, \\ x'(\eta, \beta, \gamma) = 0 \end{cases}$$

has a nontrivial solution if and only if the determinant of system is zero. After computations we have

$$\cosh \sqrt{3}\eta - 4 \cos \frac{\sqrt{3}}{2}\eta \cosh^3 \frac{\eta}{2} + 3 \cosh \eta - 4\sqrt{3} \sin \frac{\sqrt{3}}{2}\eta \sinh^3 \frac{\eta}{2} = 0.$$

It easy to show that this equation has infinitely many zeros. The first ten values of η_i and (β, γ) which relate to η_i, are given in the Table 2.

Table 2. The initial values of solutions of (15) which have a zero of 4th order at $t = 0$ and a double zero at $t = \eta_i$.

η_i	η_i value	$x^{(4)}(0) = \beta$	$x^{(5)}(0) = \gamma$	$\phi_i = \arctan\,(\frac{x_i^{(5)}(0)}{x_i^{(4)}(0)})$.
η_1	6.707630181	1	−0.9412005832	−0.755117190
η_2	10.267883954	1	−1.0102600554	−0.790501964
η_3	13.90744663	1	−0.9983471320	−0.784571046
η_4	17.53312383	1	−1.0002699913	−0.785533141
η_5	21.16103659	1	−0.9999559961	−0.785376161
η_6	24.7885841	1	−1.0000071744	−0.785401751
η_7	28.41619121	1	−0.9999988303	−0.785397579
η_8	32.043788577	1	−1.0000001906	−0.785398259
η_9	35.67138753	1	−0.9999999689	−0.785398148
η_{10}	39.29898622	1	−1.000000005	−0.785398166

All solutions with such initial conditions (β, γ) tend to a solution $x^*(t)$

$$x^*(t) = \frac{1}{3}\left(\exp(-t) - \exp\frac{t}{2}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\exp\left(-\frac{t}{2}\right)\sin\frac{\sqrt{3}}{2}t\right),$$

which satisfy initial conditions $\beta = 1$, $\gamma = -1$. (See 3.2)

For arbitrary k the same is true. This can be shown by reduction to equation $x^{(6)} = x$ and the variable change $\tau = kt$.

Proposition 3.1 *If $k_2 > k_1$ then $\eta_i(k_2) < \eta_i(k_1)$ for any i.*

Proof. We give a straightforward proof based on the variable change. Consider the linear equations

$$x^{(6)} = k_2^6 x, \tag{17}$$

and

$$x^{(6)} = k_1^6 x, \tag{18}$$

where $k_2 > k_1$. By the variable change $\tau = (\frac{k_1}{k_2})t$ the equation (18) becomes

$$\frac{d^6x}{d\tau^6} = \left(\frac{k_2}{k_1}\right)^6 \frac{d^6x}{dt^6} = k_2^6 x(t) = k_2^6\, x(\frac{k_2}{k_1}\tau)$$

or

$$\frac{d^6X}{d\tau^6} = X(\tau), \quad X(\tau) = x(\frac{k_2}{k_1}\tau)$$

and hence

$$\eta_i(k_2) = \frac{k_1}{k_2}\eta_i(k_1). \tag{19}$$

□

Consider solutions $x_i(t)$ of the problem (11), (13), (14), associated with points $\eta_i(k)$. These solutions x_i are uniquely defined (up to multiplication by a constant) by the number $i - 1$ of internal (with respect to the interval $(0, \eta_i)$ zeros as well as by the angle

$$\phi_i = \arctan\,(\frac{x_i^{(5)}(0)}{x_i^{(4)}(0)}).$$

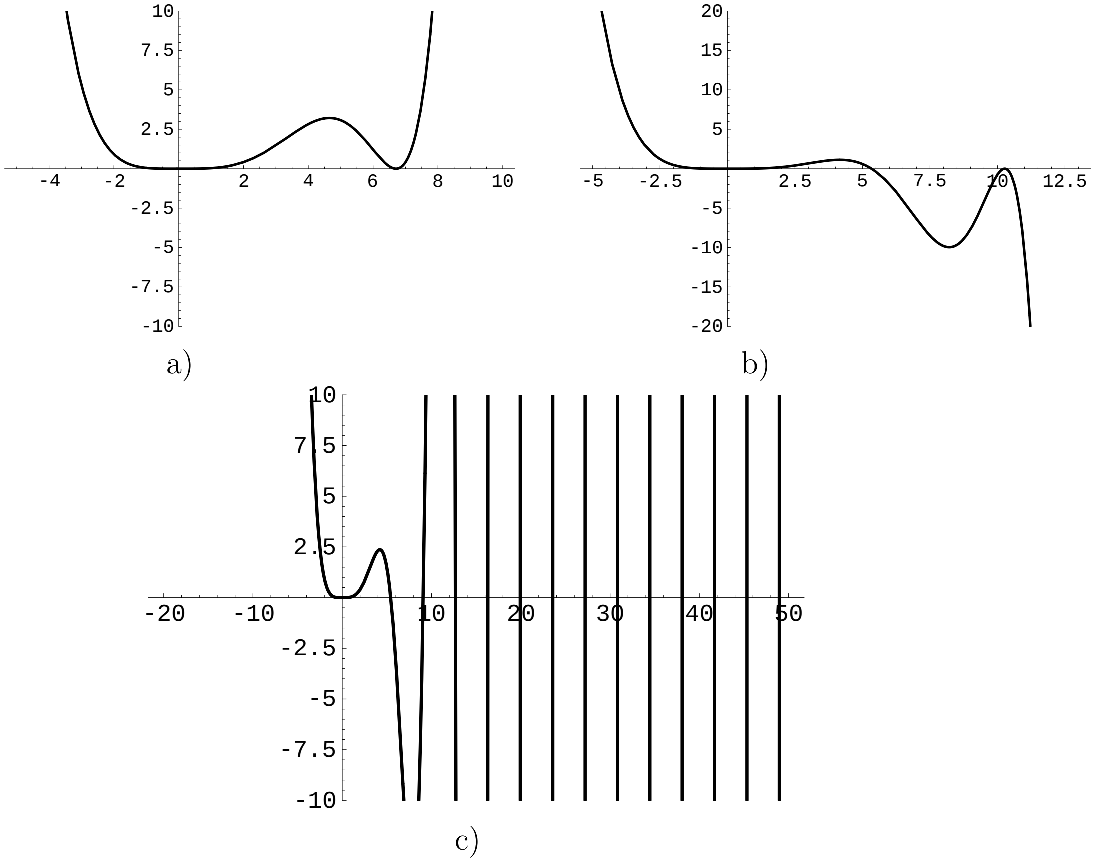


Figure 3.2: a) The $(4,2)$-solution $x(t)$ of (15) which relates to η_1; b) The $(4,2)$-solution $x(t)$ of (15) which relates to η_2; c) The solution of (15) $x^*(t)$.

It can be shown (like in the work [5] for fourth order linear two-termed equations) that both sequences $\{\eta_i\}$ and $\{\phi_i\}$ are ordered as

$$\frac{3\pi}{2} < \phi_2 < \ldots < \phi_{2k} < \ldots < \phi_{2n+1} < \ldots < \phi_1 < 2\pi$$

and

$$0 < \eta_1 < \eta_2 < \ldots .$$

We get additionally by computing derivatives of $(4,2)-$solutions that

$$\tan\phi_i(k_2) = (\frac{k_1}{k_2})\tan\phi_i(k_1). \tag{20}$$

Evidently both monotone sequences $\{\phi_{2i-1}\}$ and $\{\phi_{2i}\}$ have limits, say, ϕ_{odd} and ϕ_{even} respectively. Computations (see Table 2) show that in the case of the equation (4) one has that $\phi_{odd} = \phi_{even} = -\frac{\pi}{4}$.

4 Final remarks

4.1 Equation (1)

It seems that behavior of $(3, 3)$-solutions is likely to that of $(2, 2)$-solutions in the case of the 4th order equation (7) with $p(t) > 0$. There are only simple zeros between double zeros for equation (7). It seems (but not yet proved) that there are only simple zeros between triple zeros for respective solutions of equation (1). We know from the work by Hunt ([1]) that the first conjugate point to $t = 0$ is η_1, the first triple zero after $t = 0$. We don't know however either points η_i (next triple zeros) are conjugate points in the sense of Definition 2.1.

4.2 Equation (2)

More can be said about $(4, 2)$-solutions of the equation (2). Lemmas similar to Lemma 2.1 and Lemma 2.2 in [3] are valid for equation (2).

Lemma 4.1 *If $x(t)$ is a solution of (2) and the values of $x^{(i)}$, $i = 0, \ldots, 5$ are non-negative (but not all zero) for $t = a$, then the functions $x^{(i)}(t)$, $i = 0, \ldots, 5$ are positive for $t > a$.*

Proof. We obtain the case of $x'(a) > 0$.

First of all we show that $x'(t) > 0$, $t > a$.

We assume that there exists a such $t_0 > a$ that $x'(t_0) = 0$ and $x'(t) > 0$, if $t \in [a; t_0)$.

Then $\exists t_1 \in [a; t_0)$, $x''(t_1) < 0 \Rightarrow \exists t_2 \in [a; t_1)$, $x^{(3)}(t_2) < 0 \Rightarrow \exists t_3 \in [a; t_2)$, $x^{(4)}(t_3) < 0$ $\Rightarrow \exists t_4 \in [a; t_3)$, $x^{(5)}(t_4) < 0 \Rightarrow \exists t_5 \in [a; t_4)$, $x^{(6)}(t_5) < 0$.

Since $x^{(6)} = p(t)x$, where $p(t)$ is positive valued function and $x^{(6)}(t_5) < 0$, therefore $x(t_5) < 0$, but $x(a) \geq 0$, therefore $\exists t_6 \in [a; t_5) \subset [a; t_0)$, $x'(t_6) < 0$. A contradiction.

Now we show that $x(t) > 0$, $t > a$.

Assume that $\exists t_7 > a$, $x(t_7) < 0 \Rightarrow \exists t_8 \in [a; t_7)$, $x'(t_8) < 0$. A contradiction with the fact that $x'(t) > 0$, $t > a$ is obtained.

Since $x^{(6)} = p(t)x$, where $p(t)$ is positive valued function and $x(t) > 0$, $t > a$, therefore $x^{(6)}(t) > 0$, $t > a$.

If $x^{(6)}(t) > 0$, $t > a$, and $x^{(i)}(a) \geq 0$, $i = 5, 4, 3, 2$, than $x^{(5)}(t) > 0$, $\Rightarrow x^{(4)}(t) > 0$, $x^{(3)}(t) > 0$, $x^{(2)}(t) > 0$, $t > a$.

In other cases, if it is given that $x^{(i)}(a) > 0$, $i = 0, 2, 3, 4, 5$, the proofs is similar.□

Lemma 4.2 *If $x(t)$ is a solution of (2) and $x^{(i)}(a) \geq 0$, $i = 0, 2, 4$, $x^{(i)}(a) \leq 0$, $i = 1, 3, 5$, then the functions $x^{(i)}(t)$, $i = 0, 2, 4$, are positive and the functions $x^{(i)}(t)$, $i = 1, 3, 5$, are negative for $t < a$.*

Proof. The proof is based on the result of Lemma 4.1, by the variable change $\tau = -t$.
□

Lemma 4.3 *If $u(t)$ and $v(t)$ are two different solutions of (2) and $u^{(i)}(a) \geq v^{(i)}(a)$, $i = 0, \ldots, 5$ then $u^{(i)}(t) > v^{(i)}(t)$, $i = 0, \ldots, 5$ for $t > a$.*

Proof. The result follow from Lemma 4.1, if we obtain the function $w(t) = u(t) - v(t)$. This function is a solution of (2) and $w^{(i)}$, $i = 0, \dots, 5$ are non-negative (but not all zero). □

Suppose that there exist $(4,2)$-solutions $x_i(t)$ for the equation (2). Recall that $(4,2)$-solutions of the equation $x^{(6)} = x$ are given in (16), where (α, β) is from Table 2. Introduce the angles

$$\phi_i = \arctan\,(\frac{x_i^{(5)}(0)}{x_i^{(4)}(0)}).$$

Proposition 4.1 *There exist limits ϕ_* and ϕ^* of the sequences $\{\phi_{2k}\}$ and $\{\phi_{2k+1}\}$.*

Proof. It can be shown using Lemma 4.3 that both sequences $\{\phi_{2i}\}$ and $\{\phi_{2i+1}\}$ are monotone, the first one is increasing and the second one is decreasing. Then there exist limits ϕ_{even} and ϕ_{odd}. Evidently $\phi_{even} \leq \phi_{odd}$. We have not an example showing that strict inequality is also possible. □

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Т. Гарбуза. О решениях линейных дифференциальных уравнений 6-го порядка.

Аннотация. Рассматриваются линейные уравнения шестого порядка вида $x^{(6)} = -p(t)x$ и $x^{(6)} = p(t)x$, где $p(t)$ функция непрерывная положительная. Изучаются осцилляционные свойства решений данных уравнений.

УДК 517.927

T. Garbuza. Par 6-sestās kārtas diferenciālvienādojumu atrisinājumiem.

Anotācija. Tiek apskatīti 6−tās kārtas lineārie diferenciālvienādojumi $x^{(6)} = -p(t)x$ un $x^{(6)} = p(t)x$, kur $p(t)$ ir nepārtraukta pozitīva funkcija. Tiek pētītas šo diferenciālvienādojumu atrisinājumu oscilācijas īpašības.

Daugavpils University
Department of Natural Sciences
and Mathematics
Daugavpils, Parades str. 1
garbuza@inbox.lv

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