On solutions of the Liénard type equation

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Summary. We provide conditions on the functions $f(x)$ and $g(x)$, which ensure the existence of "small" and "large" amplitude periodic solutions to the equation $x'' + f(x)x'^2 + g(x) = 0$. Solvability of the Neumann boundary value problem is considered also.

MSC: 34B15, 34C25

1 Introduction

Intensive literature is devoted to investigation of the Liénard equation

$$x'' + f(x)x' + g(x) = 0 \tag{1}$$

due to its importance in applications. Existence (also nonexistence) of periodic solutions is the main subject of investigations. This depends of course on properties of functions f and g. Burton points out that phase portraits for (1) are well known if the function $f(x)$ is supposed to be positive and $g(x)$ is assumed to be of odd type, that is, $xg(x) > 0$ ([2], [3]). Let $F(x) = \int_0^x f(s)\,ds$. The existence of periodic solutions of (1) was studied in the work [5] provided that $F(x)$ can change sign and the amplitudes of $F(x)$ are decreasing. The function $g(x)$ was supposed to be of odd type.

On the other hand, it is known that conservative equation

$$x'' + g(x) = 0 \tag{2}$$

always has periodic solutions if the function $g(x)$ has simple zeros where $g'(x) > 0$. The equivalent system

$$\begin{cases} x' &= \quad y, \\ y' &= -g(x) \end{cases} \tag{3}$$

then has critical points of the type "center" and "small"-amplitude periodic solutions appear. In the case of $xg(x) > 0$ the only critical point is $(0;0)$ and a set (continuum) of closed curves exist in a neighborhood of the critical point. If the function $g(x)$ is negative for $x \in (p_{i+1}, +\infty)$ and positive for $x \in (-\infty, p_1)$ and there exist $i-1$ simple ($g' \neq 0$) zeros in (p_1, p_{i+1}), then equation (2) may have also "large"-amplitude solutions. The respective closed orbits go around (enclose) several critical points. For details one may consult the work [4].

From the point of view of the boundary value problems (BVP) periodic solutions with appropriate periods may satisfy some prescribed boundary conditions. We have considered the Neumann boundary conditions

$$x'(0) = 0, \quad x'(1) = 0 \tag{4}$$

in the work [1].

Recently the paper by Sabatini was published ([6]) where the equation

$$x'' + f(x)x'^2 + g(x) = 0 \tag{5}$$

was studied. Among other things the transformation was presented which turns equation (5) to the conservative form

$$u'' + h(u) = 0. \tag{6}$$

The goal of our paper is to study the equation (5) with respect to existence of periodic solutions.

In the second section we recall the results for conservative equation (2). In the third section reduction of equation (5) to (6) is considered. Results for equation (5) are established in the fourth section. Discussion follows in the fifth one and examples illustrating the results are given in the sixth section.

2 Equation $x'' + g(x) = 0$

Let $g(x)$ be a continuously differentiable function like in Fig. 2.1. Zeros of $g(x)$ are $p_1 < p_2 < p_3 < p_4 < p_5$.

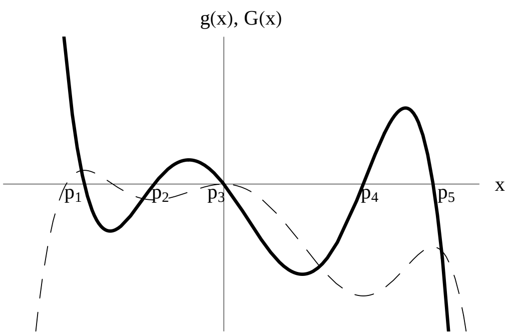

Figure 2.1 Functions $g(x)$ and $G(x)$ (the primitive)

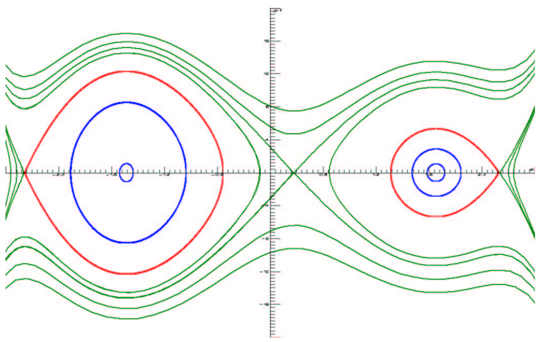

Figure 2.2 The phase plane

The equivalent system has three saddle points at $(p_1, 0)$, $(p_3, 0)$, $(p_5, 0)$ and centers at $(p_2, 0)$ and $(p_4, 0)$.

The typical phase portrait is given in Fig. 2.2. There are two sets of "small" amplitude periodic solutions located in neighborhoods of $(p_2, 0)$ and $(p_4, 0)$.

No other nontrivial periodic solutions exist if three local maxima of the primitive $G(x)$ are such that either $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$or $G(p_1) < G(p_5) < G(p_3)$.

Situation is quite different if $G(p_3)$ is less that $G(p_1)$ and $G(p_5)$. Then appear "large" amplitude periodic solutions like in Fig. 2.4.

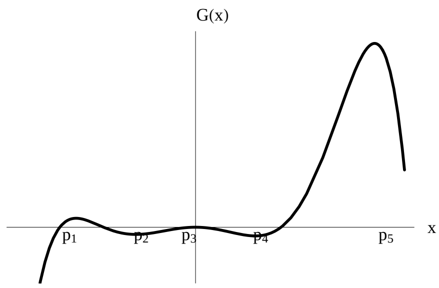

Figure 2.3 The primitive $G(x)$

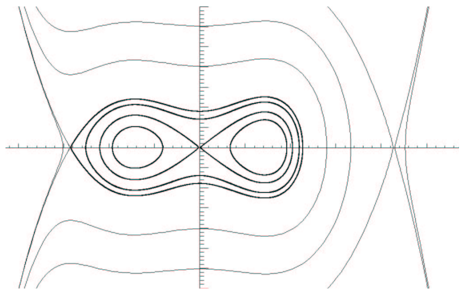
Figure 2.4 The phase plane for the case $G(p_3) < G(p_1) < G(p_5)$.

Theorem 2.1 *If $G(p_3)$ is less (strictly) than $G(p_1)$ and $G(p_5)$, then equation (2) has "large"-amplitude periodic solutions, that is, solutions with trajectories going around the critical points $(p_2; 0)$ and $(p_4; 0)$.*

Remark 2.1. If the inequalities $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$or $G(p_1) < G(p_5) < G(p_3)$ hold then "large"-amplitude periodic solutions do not exist. This can be shown. See picture 2.2, which corresponds to the case of the above mentioned.

3 Reduction of $x'' + f(x)x'^2 + g(x) = 0$ to $u'' + h(u) = 0$

Let $F(x) = \int_0^x f(s)\,ds$. Let $G(x) = \int_0^x g(s)\,ds$. The function $\Phi(x)$ was introduced in [6] by the formula

$$\Phi(x) = \int_0^x e^{F(s)}\,ds. \tag{7}$$

It is evident that $\Phi(x)$ satisfies the condition $x\Phi(x) > 0$ for $x \neq 0$. The growth rate of $\Phi(x)$ depends on properties of the primitive $F(x)$. It is important that $\Phi(x)$ is strictly monotone function for any F since $\Phi'(x) = e^{F(x)} > 0\ \forall x \in \mathbb{R}$. Then the relation

$$\Phi(x) = u \tag{8}$$

defines $u = u(x)$ and the inverse function $x = x(u)$ exists. We will use these functions defined for various F throughout in our considerations.

Our further study employs the following basic result from [6].

Lemma 3.1 ([6], Lemma 1) *The function $x(t)$ is a solution to (5) if and only if $u(t) = \Phi(x(t))$ is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{9}$$

Denote $H(u) = \int_0^u g(x(s))e^{F(x(s))}\,ds$. The existence of periodic solutions and the existence of solutions to the Neumann BVP depends entirely on properties of the primitive H.

Let us state some easy assertions about equation (5) and the equivalent system

$$\begin{cases} x' &= y, \\ y' &= -f(x)y^2 - g(x) \end{cases} \tag{10}$$

Proposition 3.1 *Critical points and their character are the same for systems (3) and (10).*

Proof. Critical points of both systems are the points $(x_i, 0)$, where x_i are zeros of $g(x)$. Points $(p_1, 0)$, $(p_3, 0)$, $(p_5, 0)$ are saddle points and $(p_2, 0)$ and $(p_4, 0)$ are the centers. Consider linearized at a point $(p_i, 0)$ system (3)

$$\begin{cases} \xi' &= \eta, \\ \eta' &= -g_x(p_i)\xi \end{cases} \tag{11}$$

where p_i is a zero of $g(x)$. Consider also linearized at a point $(p_i, 0)$ system (10)

$$\begin{cases} \alpha' &= \beta, \\ \beta' &= -[f_x(p_i)y + g_x(p_i)]|_{(p_i,0)}\alpha + [2f(x)y]|_{(p_i,0)}\beta \\ &= -g_x(p_i)\alpha \end{cases} \tag{12}$$

$$\begin{cases} \alpha' &= \beta, \\ \beta' &= -[f_x((p_i,0))y^2 + g_x((p_i,0))]\alpha + [2f((p_i,0))y]\beta \end{cases} \tag{13}$$

$$\begin{cases} \alpha' &= \beta, \\ \beta' &= -g_x((p_i;0))\alpha \end{cases} \tag{14}$$

Systems (11) and (14) up to notation are the same. □

Consider a system

$$\begin{cases} u' &= v, \\ v' &= -g(x(u))e^{F(x(u))} \end{cases} \tag{15}$$

equivalent to equation (9).

Proposition 3.2 *Critical points $(x, 0)$ and $(u(x), 0)$ of systems (3) and (15) are in 1-to-1 correspondence and their characters are the same.*

Proof. Let us show that critical points $(x, 0)$ of system (3) turn to critical points $(u(x), 0)$ of system (15). Then by Proposition 3.1 critical points $(x, 0)$ of system (3) turn to critical points $(u(x), 0)$ of system (15).

Proposition 3.3 *Periodic solutions $x(t)$ of equation (5) turn to periodic solutions $u(t) = \Phi(x(t))$ by transformation (8).*

Proposition 3.4 *Homoclinic solutions of (5) turn to homoclinic solutions of equation (9) by transformation (8).*

Proposition 3.5 *Let p be a zero of $g(x)$. The equality*

$$g_x(p) = g_u(x(u))e^{F(x(u))}|_{u=p} \tag{16}$$

is valid.

Proof. By calculation of the derivative.

4 The result

Consider equation (9). This equation is conservative and Theorem 2.1 applies. Denote $H(u) = \int_0^u g(x(s))e^{F(x(s))}\,ds$. The function $H(u)$ has the same structure as $G(x)$ that is, it has exactly 3 points of maxima and 2 minimum points. Moreover, $H(u)$ has three local maxima at the points $u(p_1)$, $u(p_3)$ and $u(p_5)$, where u is as in (8), and two local minima at the points $u(p_2)$ and $u(p_4)$.

Theorem 4.1 *Let the inequalities hold:*

$$\begin{array}{l} H(u(p_3)) < H(u(p_1)) \\ H(u(p_3)) < H(u(p_5)) \end{array} \tag{17}$$

Then equation (5) has "large"-amplitude periodic solutions.

Proof. By application theorem 2.1 to equation (5) and using Proposition (3.3).

5 Case $f(x) = 2kx$

Then $f(x) = 2kx$ and $F(x) = kx^2$, $u = \Phi(x) = \int_0^x e^{ks^2}\,ds$. Equation (9) takes the form

$$u'' + g(x(u))e^{kx^2(u)} = 0. \tag{18}$$

Consider the equation (5) with $f(x) = 2kx$ and equation (2). Suppose that $G(x)$ is such that $G(p_1) > G(p_3) > G(p_5)$. Then by Theorem 2.1 and Remark after "large"-amplitude periodic solutions of (3) do not exist. Then equation (18) and, consequently, equation (5) do not have "large"-amplitude periodic solutions. For large enough $k > 0$ the shape of $H(u)$ changes so that the inequalities (17) hold. Then "large"-amplitude periodic solutions appear in (18), and, consequently, in equation (5). Further growth of $k > 0$ results in the fact that $H(u(p_1)) < H(u(p_3)) < H(u(p_5))$. Then "large"-amplitude solutions disappear.

The explanation is the following.

1) k is "small", the inequalities $H(u(p_1)) > H(u(p_3)) > H(u(p_5))$ hold.

2) k is "middle", then the inequalities (17) hold (as a result of multiplication $g(x)$ by e^{kx^2} in (18)).

3) k is "large", then the inequalities $H(u(p_1)) < H(u(p_3)) < H(u(p_5))$ hold.

6 Example

Let consider equation (2), where

$$g(x) = -0.108x + 0.831x^2 - 2.23x^3 + 2.5x^4 - x^5. \tag{19}$$

The function (19) has exactly 5 simple zeros $p_1 = 0$; $p_2 = 0.3$; $p_3 = 0.5$; $p_4 = 0.8$; $p_5 = 0.9$. The equivalent two-dimensional system (3) has 3 critical points of the type "saddle"

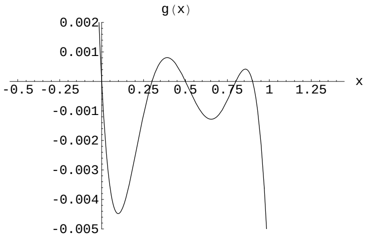

Figure 6.1 Function $g(x)$

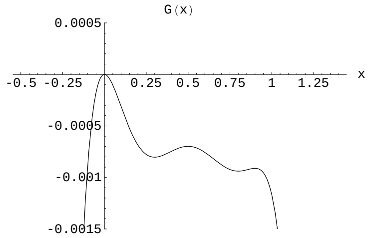

Figure 6.2 Function $G(x)$

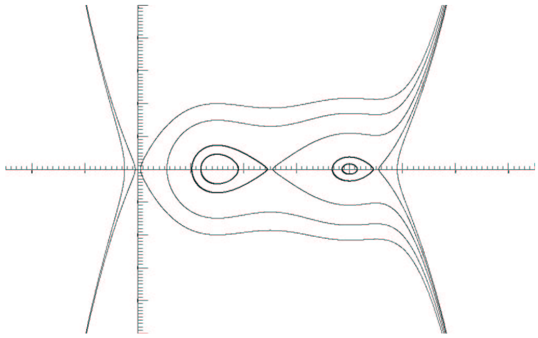

Figure 6.3 The phase plane

and 2 critical points of the type "center". Respectively the function $G(x) = \int_0^x g(s)\,ds$ has 3 local maxima and 2 local minima as its shown in Fig. 6.1 and 6.2.

How we can see "large"-amplitude periodic solutions of in this case do not exist.

Let consider equation (5) with (19) and $f(x) = 2kx$.

1) If $k = 5$, then there are only "small"-amplitude periodic solutions.

p_i	u_i
0	0
0.3	0.351787
0.5	0.816757
0.8	3.83505
0.9	7.66983

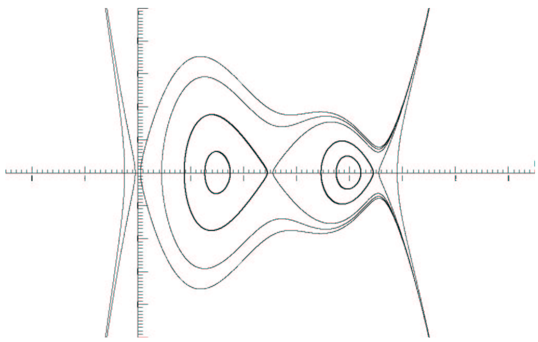

Figure 6.4 The phase plane

2) If $k = 13$, then there are "small"-amplitude and "large"-amplitude periodic solu-

tions.

p_i	u_i
0	0
0.3	0.472704
0.5	2.47552
0.8	212.627
0.9	1690.25

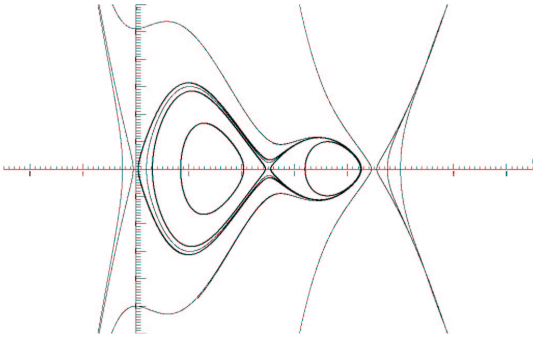

Figure 6.5 The phase plane

3) If $k = 20$, then there are only "small"-amplitude periodic solutions.

p_i	u_i
0	0
0.3	0.639123
0.5	8.58608
0.8	11828.1
0.9	311828

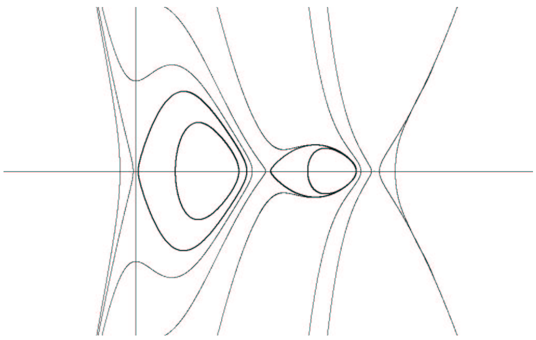

Figure 6.6 The phase plane

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С. Атслега. О решениях уравнения типа Льенара.
Аннотация. Приводятся условия на функции $f(x)$ и $g(x)$, достаточные для существования решений уравнения $x'' + f(x)x'^2 + g(x) = 0$ с “малыми” и “большими” амплитудами. Рассматривается также разрешимость задачи Неймана на фиксированном интервале.
УДК 517.927

S. Atslega. Rezultāti par Ljenara tipa vienādojuma atrisinājumiem.
Anotācija. Tiek apskatīts Ljenara tipa diferenciālvienādojums $x'' + f(x)x'^2 + g(x) = 0$. Doti pietiekamie nosacījumi ”mazo” un ”lielo” amplitūdu atrisinājumu eksistencei funkciju f un g terminos. Papildus tiek pētīta Neimana problēma uz fiksēta intervālā.

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