# On solutions of the fourth-order nonlinear boundary value problems

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**Summary.** We consider a two-point boundary value problem for the fourth-order non-autonomous Emden-Fowler type equation using the quasilinearization process. We reduce the given nonlinear equation to a some quasi-linear one with a non-resonant linear part so that both equations are equivalent in some bounded domain. We use a fact that modified quasi-linear problem has a solution of definite type, which corresponds to the type of the linear part. If a solution of the quasi-linear problem is located in the domain of equivalence, then the original problem has a solution. If quasilinearization is possible for essentially different linear parts, then the original problem has multiple solutions.

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## 1 Introduction

Consider two-point boundary value problem for the fourth-order nonlinear differential equation

$$x^{(4)} = q(t) \cdot |x|^p \operatorname{sgn} x, \tag{1}$$

$$x(0) = x'(0) = 0 = x(1) = x'(1), \tag{2}$$

where $p > 1, \quad t \in I := [0, 1], \; q(t) \in C(I, (0, +\infty))$.

Our aim is to obtain conditions for existence of multiple solutions. We investigate the problem (1), (2) by reducing it to multiple quasi-linear problems of different types. Suppose that equation (1) can be reduced to the quasi-linear one of the form

$$x^{(4)} - k^4 x = F(t, x), \tag{3}$$

where $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is a non-resonant linear part, function $F(t, x)$ is continuous and bounded and equations (1), (3) are equivalent in some domain

$$\Omega = \{(t, x) : \; 0 \le t \le 1, \; |x| \le N\}.$$

If a solution $x(t)$ of the quasi-linear problem (3), (2) is located in the domain of equivalence $\Omega(t,x)$ ($|x(t)| \leq N$), then this $x(t)$ also solves the original problem (1), (2). We then say that the original problem allows for quasilinearization with respect to the linear part $\big(L_4x\big)(t) := x^{(4)} - k^4 x$.

If the original problem allows for quasilinearization for different values of $k$, then in some cases we can obtain different solutions, that is the solutions with different oscillatory properties. Let us illustrate this by considering the second-order problem.

# 2 Second-order linear problems and respective linear parts

Consider the second-order Cauchy problems (for various $k$)

$$\begin{gathered} x'' + k^2 x = 0, \\ x(0) = 0, \quad x'(0) = 1. \end{gathered} \tag{4}$$

Linear parts $\big(L_2x\big)(t) := x'' + k^2x$ are non-resonant (this means that the respective homogeneous problems $x'' + k^2x = 0, \quad x(0) = 0, \quad x(1) = 0$ have only the trivial solution), if the coefficient $k$ belongs to one of the intervals

$$(0,\, \pi),\ (\pi,\, 2\pi), \ldots, (i\pi,\, (i+1)\pi), \ldots$$

Definition of an $i$-nonresonance of the linear part was given in [7], [8]. If the linear parts $x'' + k_i^2x$ and $x'' + k_j^2x$ are respectively $i$-nonresonant and $j$-nonresonant, we say then that such linear parts are essentially different. If the linear parts of the problems (4) are essentially different, then respective quasi-linear boundary value problems have solutions with different oscillatory properties.

For the values of $k$ from different intervals (the above mentioned) the solutions of the problem (4) with different oscillatory properties are obtained and shown below.

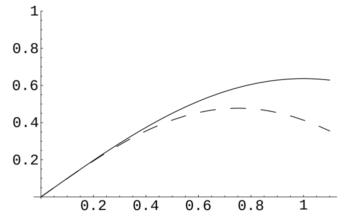


Figure 2.1. 0-nonresonance of a linear part

Figure 2.1 shows the solutions of the problem (4) for $k_1 = \frac{\pi}{2}$ and $k_2 = \frac{2\pi}{3}$. The linear parts $x'' + k_1^2x$ and $x'' + k_2^2x$ are 0-nonresonant.

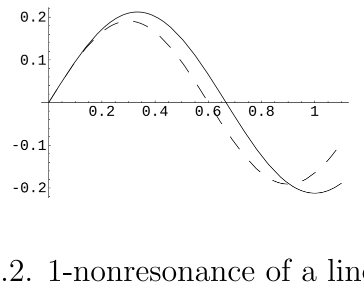


Figure 2.2. 1-nonresonance of a linear part

Figure 2.2 illustrates the solutions of the problems (4) for $k_3 = \frac{3\pi}{2}$ and $k_4 = \frac{5\pi}{3}$; each of them has exactly one zero in the interval (0, 1). The respective linear parts $x'' + k_3^2 x$ and $x'' + k_4^2 x$ are 1-nonresonant.

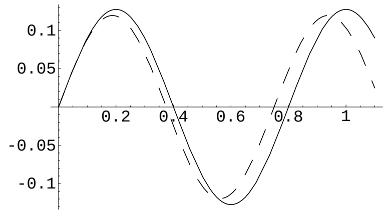


Figure 2.3. 2-nonresonance of a linear part

If $k \in (2\pi, 3\pi)$ (for instance, $k_5 = \frac{5\pi}{2}$ and $k_6 = \frac{8\pi}{3}$), then the respective solution of the problem (4) has exactly two zeros in the interval (0, 1) (see Figure 2.3) and the corresponding linear part is 2-nonresonant.

Therefore, if the values of $k$ belong to the same interval $(i\pi, (i+1)\pi)$, then the respective solutions of the problems (4) have similar oscillatory properties. If the numbers $k_i$ and $k_j$ belong to different intervals $(i\pi, (i+1)\pi)$ and $(j\pi, (j+1)\pi)$ $(i \neq j)$, then the linear parts $x'' + k_i^2 x$ and $x'' + k_j^2 x$ have different type of nonresonance (the linear parts are essentially different) and the respective solutions of the problems (4) have different oscillatory properties.

# 3 Fourth-order quasi-linear problems and types of solutions

Consider the quasi-linear problems of the form

$$\begin{gathered} x^{(4)} - k^4 x = F(t, x), \\ x(0) = x'(0) = 0 = x(1) = x'(1), \end{gathered} \tag{5}$$

where $t \in I := [0, 1]$, $F$, $F_x : I \times R \to R$ are continuous, $F$ is bounded and the following condition is satisfied for any $(t, x)$

$$k^4 + \frac{\partial F(t, x)}{\partial x} > 0. \tag{6}$$

In our investigation we use the oscillation theory by Leighton-Nehari for the fourth-order linear differential equations [4]

$$x^{(4)} - p(t)x = 0, \qquad p(t) > 0. \tag{7}$$

We use their definition of a conjugate point and define an $i$-nonresonanse of the linear part and an $i$-type solution of the quasi-linear problem (5) (see [9], [11], [12]).

*Remark 3.1.* The conjugate points (or double zeros) in the oscillation theory for the fourth-order linear differential equations play the same role as the ordinary zeros in the oscillation theory for the second-order equations.

*Remark 3.2.* We use the following facts in our investigation:
1) for the linear equation $x^{(4)} - k^4 x = 0$ the respective conjugate points $\eta$ (to the point $t = 0$) satisfy the relation $\cos k\eta \cdot \cosh k\eta = 1$;
2) if the values of $k$ are in the form $\pi n$ $(n = 1, 2, \ldots)$, then there exist $(n-1)$ conjugate points in the interval $(0, 1)$, in other words the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is $(n-1)$-nonresonant for $k = \pi n$, $n = 1, 2, \ldots$.

Let us recall the definition of an $i$-type solution.

**Definition 3.1** *We will say that $\xi(t)$ is an $i$-type solution of the quasi-linear problem* (5)*, if for small enough $\alpha, \beta > 0$ the difference $u(t; \alpha, \beta) = x(t; \alpha, \beta) - \xi(t)$ has at most $i + 1$ zeros in the interval* $(0, 1]$ *(counting multiplicities), where $x(t; \alpha, \beta)$ is a solution of the same quasi-linear equation, which satisfies the initial conditions*

$$x(0; \alpha, \beta) = \xi(0), \quad x'(0; \alpha, \beta) = \xi'(0), \tag{8}$$

$$x''(0; \alpha, \beta) = \xi''(0) + \alpha, \quad x'''(0; \alpha, \beta) = \xi'''(0) - \beta. \tag{9}$$

We call the solution $x(t; \alpha, \beta)$ by *neighboring* solution.

The following theorem plays the basic role in our investigation.

**Theorem 3.1** *The quasi-linear problem* (5) *has an $i$-type solution, if the condition* (6) *is fulfilled and the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$ is $i$-nonresonant.*

Theorem 3.1 was proved in [9], [12].

# 4 Application

We apply Theorem 3.1 to the boundary value problem (1), (2), that is, to the problem

$$\begin{gathered} x^{(4)} = q(t) \cdot |x|^p \operatorname{sgn} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{gathered} \tag{10}$$

**Theorem 4.1** *Suppose that*

$$0 < q_1 \le q(t) \le q_2 \quad \forall t \in [0,\ 1]. \tag{11}$$

*If there exists some $k$ of the form $k = \pi i$, $(i = 1, 2, \ldots)$, which satisfies the inequality*

$$k \cdot \frac{e^k(4\sqrt{2} + 3) - 1}{4(e^k + 1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{(p-1)} \Big(\frac{q_1}{q_2}\Big)^{\frac{1}{(p-1)}} \qquad \text{for} \quad k = (2n-1)\pi \tag{12}$$

*or*

$$k \cdot \frac{e^k(4\sqrt{2} + 3) + 1}{4(e^k - 1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{(p-1)} \Big(\frac{q_1}{q_2}\Big)^{\frac{1}{(p-1)}} \qquad \text{for} \quad k = 2n\pi, \tag{13}$$

*where $\beta$ is a positive root of the equation*

$$\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}, \tag{14}$$

*then there exists an $(i-1)$-type solution of the problem* (1), (2).

**Proof.** The given nonlinear equation (1) is equivalent to the equation

$$x^{(4)} - k^4 x = q(t) \cdot |x|^p \operatorname{sgn} x - k^4 x. \tag{15}$$

Suppose that $k$ satisfies the non-resonance condition $\cos k \cdot \cosh k \neq 1$ and the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is therefore non-resonant with respect to the boundary conditions (2).

Denote the right side of the equation (15) by $f_k(t,\, x)$ and try to bound it. For a fixed $t = t^*$ we can calculate the value of the function $f_k(t,\, x)$ at the point of extremum $x_0$. Set

$$m_k(t^*) = |f_k(t^*,\, x_0)| = \Big(\frac{k^4}{p}\Big)^{\frac{p}{p-1}} \cdot (p-1) \cdot q(t^*)^{\frac{1}{1-p}}. \tag{16}$$

Choose $n_k(t^*)$ such that

$$|x| \le n_k(t^*) \Rightarrow |f_k(t^*,\, x)| \le m_k(t^*).$$

Computation gives that

$$n_k(t^*) = \Big(\frac{k^4}{q(t^*)}\Big)^{\frac{1}{p-1}} \beta, \tag{17}$$

where a constant $\beta > 1$ is described in (14). Set

$$M_k = \max\{m_k(t^*) : t^* \in [0,\, 1]\}, \quad N_k = \min\{n_k(t^*) : t^* \in [0,\, 1]\}.$$

Let us consider now the corresponding quasi-linear equation

$$x^{(4)} - k^4 x = F_k(t, x), \tag{18}$$

where $F_k(t,x) := \varphi(x) f_k(t,\, x)$ and function $\varphi(x)$ is such that $\varphi = 1$, if $|x(t)| \le N_k$, and $F_k(t,x)$ is smooth and bounded by the value of $M_k > 0$.

The modified quasi-linear problem (18), (2) can be written in the integral form

$$x(t) = \int_0^1 G_k(t,s) F_k(s, x(s))\, ds,$$

where $G_k(t,s)$ is the Green's function for the respective homogeneous problem

$$\begin{gathered} x^{(4)} - k^4 x = 0, \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{gathered} \tag{19}$$

We have constructed the Green's function $G_k(t,s)$ and have obtained the estimate for it in [9], [11]. Then a solution of the quasi-linear problem (18), (2) satisfies

$$\big|x(t)\big| \le \Gamma_k \cdot M_k$$

($\Gamma_k$ is an estimate of the Green's function). If moreover the inequality

$$\Gamma_k \cdot M_k \, < \, N_k \tag{20}$$

holds, then equations (1) and (18) are equivalent in the domain

$$\Omega_k = \{(t,x):\ 0 \le t \le 1,\ |x| < N_k\}.$$

Notice that in this domain of equivalence $\Omega_k$ the function $F_k(t,x)$ is continuously differentiable and the condition (6) is fulfilled (i.e. $k^4 + \dfrac{\partial F(t,\,x)}{\partial x} > 0$). So it follows from Theorem 3.1 that if the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$ is $i$-nonresonant, then the quasi-linear problem (18), (2) has an $i$-type solution, if moreover the inequality (20) holds, then the original problem (1), (2) also has an $i$-type solution.

Consider the inequality (20) and assume that $q(t)$ satisfies (11). Since $p > 1$, then

$$\begin{gathered}\max_{t^* \in [0,\,1]} m_k(t^*) = \left(\tfrac{k^4}{p}\right)^{\frac{p}{p-1}} \cdot (p-1) \cdot q_1^{\frac{1}{1-p}}, \\ \min_{t^* \in [0,\,1]} n_k(t^*) = \left(\tfrac{k^4}{q_2}\right)^{\frac{1}{p-1}} \beta.\end{gathered} \tag{21}$$

Therefore the inequality (20) takes the form

$$k^4 \cdot \Gamma_k < \beta \cdot \frac{p^{\frac{p}{p-1}}}{(p-1)} \left(\frac{q_1}{q_2}\right)^{\frac{1}{(p-1)}}. \tag{22}$$

Let us consider values $k$ of the form $k = \pi i\,(i = 1,\, 2\ \ldots)$. For such $k$ the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$ is $(i-1)$-nonresonant and the Green's function $G_k(t,s)$ satisfies either the estimate

$$\big|G_k(t,s)\big| < \frac{e^k(4\sqrt{2}+3)-1}{4k^3(e^k+1)} =: \Gamma_1(k), \qquad \text{if} \quad k = (2n-1)\pi \tag{23}$$

or

$$\big|G_k(t,s)\big| < \frac{e^k(4\sqrt{2}+3)+1}{4k^3(e^k-1)} =: \Gamma_2(k), \qquad \text{if} \quad k = 2n\pi. \tag{24}$$

It follows from (22), (23), (24) that the inequality (20) reduces respectively either to (12) or (13). Therefore if there exists some $k$ in the form $k = \pi i$, $(i = 1, 2, \ldots)$, which satisfies an inequality (12) or (13), then there exists an $(i-1)$- type solution of the given problem (1), (2). The proof is complete.

**Corollary 4.1** *If there exist $k = \pi i$, $i = 1, 2, \ldots, m$, which satisfy the inequalities* (12), (13)*, then there exist at least $m$ solutions of different types to the problem* (1), (2).

In Appendix we provide a table of the obtained results of calculations. For certain values of $p$ and $\dfrac{q_1}{q_2}$ the numbers $k$ in the form $k = \pi i$, $i = 1, 2, \ldots$ are given, which satisfy the inequalities (12), (13).

# 5 Example

Consider the fourth-order nonlinear boundary value problem

$$\begin{gathered}x^{(4)} = 50(81 + \sin \tfrac{\pi}{2} t)|x|^{\frac{9}{8}} \operatorname{sgn} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1).\end{gathered} \tag{25}$$

It is a special case of the problem (1), when $p = \dfrac{9}{8}$ and $q(t) = 50(81 + \sin \frac{\pi}{2}t)$. Since $\min\limits_{[0,1]} q(t) = 4050$ and $\max\limits_{[0,1]} q(t) = 4100$ then the quotient $\dfrac{q_1}{q_2} = \dfrac{81}{82}$. So in accordance with calculations (see Table 1 in Appendix) and Corollary 3.1 there exist at least four solutions of different types to the given problem (25). We have computed them.

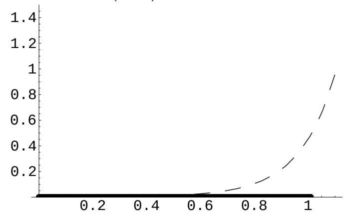


Figure 5.1. 0-type solution

The solid line in Figure 5.1 indicates a trivial solution of the problem (25) and the dashed line relates to one of the corresponding neighboring solutions (see Definition 3.1). All other neighboring solutions are such that the difference between neighboring solution and the trivial one has no conjugate points (double zeros) in the interval (0, 1), therefore the trivial solution is a 0-type solution. Figure 5.2 shows another solution of the problem

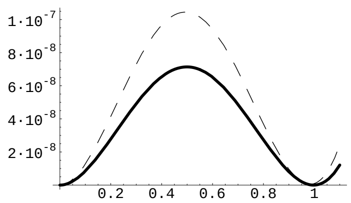


Figure 5.2. 1-type solution

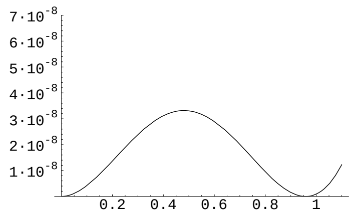


Figure 5.3. The difference between neighboring solution and 1-type sol.

(25) in solid (one of the corresponding neighboring solutions is shown in dashed). This solution is an 1-type solution because the difference between neighboring solution and it (see Figure 5.3) has exactly one conjugate point in (0, 1) $\eta = 0,959862$. The initial data of the 1-type solution are $x''(0) = 0,000002$, $x'''(0) = -0,000009223$.

Figure 5.4 illustrates a 2-type solution of the problem (25). The initial data of this solution are $x''(0) = 51$, $x'''(0) = -395,08258$. The graphs of the respective neighboring solutions is difficult to show, because two lines almost coincide. Nevertheless, the differences between some neighboring solutions and this solution are depicted in Figure 5.5 and Figure 5.6. There exist exactly two conjugate points in (0, 1): $\eta_1 = 0,586519$, $\eta_2 = 0,972921$.

Figure 5.7 illustrates a 3-type solution of the problem (25). The initial data of this solution are $x''(0) = 5100000$, $x'''(0) = -55374924,809$. The differences between respective neighboring solutions and this solution are depicted in Figure 5.8, Figure 5.9 and Figure 5.10. There exist exactly three conjugate points in (0, 1): $\eta_1 = 0,418951$, $\eta_2 = 0,695755$, $\eta_3 = 0,973976$.

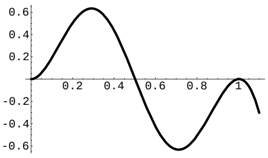


Figure 5.4.2-type solution

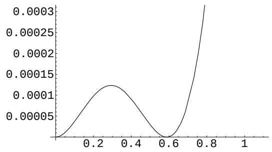


Figure 5.5. The difference between neighboring solution and 2-type solution, $\alpha_1 = 0,01$, $\beta_1 = 0,07823$

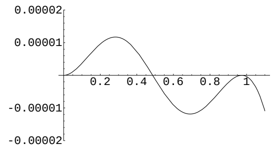


Figure 5.6.The difference between neighboring solution and 2-type solution, $\alpha_2 = 0,001$, $\beta_2 = 0,0079747$

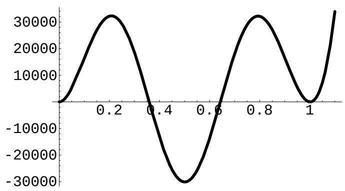


Figure 5.7. 3-type solution

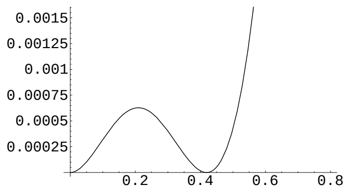


Figure 5.8. The difference between neighboring solution and 3-type sol., $\alpha_1 = 0,1$, $\beta_1 = 1,0974$

# 6 Remarks about the sublinear case

It should be said that the same type arguments are valid in the case of the Emden - Fowler type equation (10), where $0 < p < 1$.

The difficulty is that the right side of this equation is not differentiable at $x = 0$. So the type of the trivial solution $x \equiv 0$ is indefinite. If we ignore it, all other possible solutions (nontrivial ones) have definite type which under certain conditions can be revealed by the described above quasi-linearization process. This process is possible if some relations of the type (12), (13) hold. The respective calculations were carried out and the results are represented in Table 2.

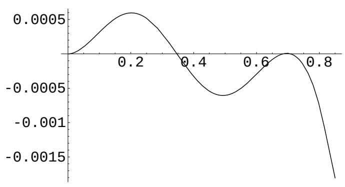


Figure 5.9. The difference between neighboring solution and 3-type sol., $\alpha_2 = 0,1$, $\beta_2 = 1,1187$

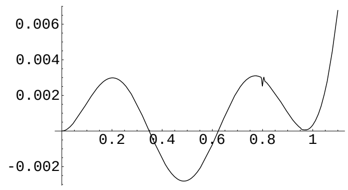


Figure 5.10.The difference between neighboring solution and 3-type sol., $\alpha_3 = 0,5$, $\beta_3 = 5,5881$

# 7 Appendix

**Table 1.** Superlinear case $p > 1$.

| | | | |
|---|---|---|---|
| $p = \dfrac{5}{4}$ | $\beta \approx 1.2813$ | $\frac{q_1}{q_2} \geq \frac{29}{30}$ | $k = \pi;\ k = 2\pi$ |
| $p = \dfrac{6}{5}$ | $\beta \approx 1.2884$ | $\frac{q_1}{q_2} \geq \frac{15}{16}$ | $k = \pi;\ k = 2\pi$ |
| $p = \dfrac{7}{6}$ | $\beta \approx 1.2933$ | $\frac{q_1}{q_2} \geq \frac{12}{13}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{53}{54}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| $p = \dfrac{8}{7}$ | $\beta \approx 1.2969$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{27}{28}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| $p = \dfrac{9}{8}$ | $\beta \approx 1.2998$ | $\frac{q_1}{q_2} \geq \frac{10}{11}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{21}{22}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{81}{82}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$ |
| $p = \dfrac{10}{9}$ | $\beta \approx 1.3019$ | $\frac{q_1}{q_2} \geq \frac{10}{11}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{18}{19}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{43}{44}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$ |
| $p = \dfrac{11}{10}$ | $\beta \approx 1.3038$ | $\frac{q_1}{q_2} \geq \frac{10}{11}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{17}{18}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{32}{33}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{111}{112}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$ |
| $p = \dfrac{12}{11}$ | $\beta \approx 1.3053$ | $\frac{q_1}{q_2} \geq \frac{10}{11}$ | $k = \pi;\ k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{16}{17}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{27}{28}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{60}{61}$ | $k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$ |

**Table 2.** Sublinear case $0 < p < 1$.

| | | | |
|---|---|---|---|
| $p = \frac{4}{5}$ | $\beta \approx 1.3632$ | $\frac{q_1}{q_2} \geq \frac{25}{26}$ | $k = \pi$; $k = 2\pi$ |
| $p = \frac{5}{6}$ | $\beta \approx 1.3554$ | $\frac{q_1}{q_2} \geq \frac{15}{16}$ | $k = \pi$; $k = 2\pi$ |
| $p = \frac{6}{7}$ | $\beta \approx 1.3499$ | $\frac{q_1}{q_2} \geq \frac{13}{14}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{45}{46}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| $p = \frac{7}{8}$ | $\beta \approx 1.3461$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{27}{28}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| $p = \frac{8}{9}$ | $\beta \approx 1.3431$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{21}{22}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{69}{70}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$ |
| $p = \frac{9}{10}$ | $\beta \approx 1.3407$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{19}{20}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{42}{43}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$ |
| $p = \frac{10}{11}$ | $\beta \approx 1.3388$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{18}{19}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{32}{33}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{94}{95}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$; $k = 5\pi$ |
| $p = \frac{11}{12}$ | $\beta \approx 1.3373$ | $\frac{q_1}{q_2} \geq \frac{11}{12}$ | $k = \pi$; $k = 2\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{17}{18}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{28}{29}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$ |
| | | $\frac{q_1}{q_2} \geq \frac{58}{59}$ | $k = \pi$; $k = 2\pi$; $k = 3\pi$; $k = 4\pi$; $k = 5\pi$ |

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**И. Ермаченко, Ф. Садырбаев. О решениях нелинейной краевой задачи четвертого порядка**.

**Аннотация.** Процесс квазилинеаризации применяется к изучению двухточечной нелинейной краевой задачи для неавтономного уравнения Эмдена - Фаулера четвертого порядка. Данное нелинейное уравнение сводится к квазилинейному уравнению с нерезонансной линейной частью таким образом, что оба уравнения эквивалентны в некоторой ограниченной области. Используется примечательный факт, что модифицированная нелинейная задача имеет решение определенного типа, соответствующего типу линейной части. Если решение квазилинейной задачи принадлежит области эквивалентности, то оно же является и решением исходной задачи. Если описанный процесс квазилинеаризации возможен для существенно линейных частей, то исходная задача имеет несколько решений.

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**I. Jermačenko, F. Sadirbajevs. Par ceturtās kārtas nelineāro robežproblēmu atrisinājumiem.**

**Anotācija.** Tiek apskatītas divpunktu nelineārās robežproblēmas neautonomiem ceturtās kārtas Emdena-Faulera tipa vienādojumiem. Doto nelineāro diferenciālvienādojumu reducē uz kādu kvazi-lineāro vienādojumu ar nerezonantu lineāru daļu tā, lai abi vienādojumi būtu ekvivalenti kādā ierobežotā apgabalā. Mēs izmantojam faktu, ka iegūtai kvazi-lineārai problēmai ir noteiktā tipa atrisinājums, kurš atkarīgs no lineāras daļas tipa. Ja kvazi-lineārās problēmas atrisinājums atrodas ekvivalences apgabalā, tad dotā nelineārā problēma ir atrisināma. Ja kvazilineārizācija ir iespējama ar būtiski dažādām lineārām daļām, tad dotajai problēmai ir vairāki atrisinājumi.

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