On Fučik spectra for the third and fourth order equations [1]

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Summary. We construct Fučik spectra for some specific differential equations of the third and fourth order. These spectra differ essentially from the known ones.

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1 Introduction

In this work we study Fučik spectra for some third and fourth order equations with piece-wise linear right sides.

Investigations of Fučik spectra have started in sixtieth of XX century [1]. A number of authors have studied the specific cases. Let us mention the cases of the Dirichlet [1], the Sturm-Liouville [5] and the periodic boundary conditions. There are some papers on higher order equations. Habets and Gaudenzi have studied the third order problem with the boundary conditions $x(0) = x'(0) = 0 = x(1)$ in the work [3], where many useful references on the subject can be found. Fučik spectra for the fourth order equations were considered by Kreiči [2] and Pope [4].

The paper is organized as follows. In Section 2 we study the third order problem with the boundary conditions $x(0) = x'(0) = 0 = x'(1)$. In Section 3 we present results on the Fučik spectrum for the boundary conditions $x(0) = x'(0) = 0 = x(1)$. These are the main results of the work. Connections between those spectra are discussed in Section 4.

2 The third order problem with the boundary conditions $\mathbf{x(0) = x'(0) = 0 = x'(1)}$

Consider the equation

$$x''' = -\mu^2 x'^+ + \lambda^2 x'^-, \quad \mu, \lambda \geq 0, \tag{2.1}$$

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$$x'^{+} = \max\{x', 0\}, \quad x'^{-} = \max\{-x', 0\},$$

with the boundary conditions

$$x(0) = x'(0) = 0 = x'(1). \tag{2.2}$$

Definition 1 *The Fučik spectrum is a set of points* (λ, μ) *such that the problem* (2.1), (2.2) *has nontrivial solutions.*

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$ according to the number of zeroes of the derivative of a solution to the problem (2.1), (2.2) in the interval $(0, 1)$.

Proposition 1 *The Fučik spectrum consists of the set of curves*
$F_i^+ = \{(\lambda, \mu)|\ x''(0) > 0,$ *the derivative of the nontrivial solution of the problem*$(2.1), (2.2)$ $x'(t)$ *has exactly* i *zeroes in* $(0, 1)\}$;
$F_i^- = \{(\lambda, \mu)|\ x''(0) < 0,$ *the derivative of the nontrivial solution of the problem*$(2.1), (2.2)$ $x'(t)$ *has exactly* i *zeroes in* $(0, 1)\}$.

Theorem 2.1 *The Fučik spectrum for the problem* (2.1), (2.2) *consists of the branches given by*

$$\begin{aligned}
F_0^+ &= \Big\{(\lambda, \pi)\Big\}, \\
F_0^- &= \Big\{(\pi, \mu)\Big\}, \\
F_{2i-1}^+ &= \Big\{(\lambda, \mu)\Big|\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} = 1\Big\}, \\
F_{2i}^+ &= \Big\{(\lambda, \mu)\Big|\ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} = 1\Big\}, \\
F_{2i-1}^- &= \Big\{(\lambda, \mu)\Big|\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} = 1\Big\}, \\
F_{2i}^- &= \Big\{(\lambda, \mu)\Big|\ \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} = 1\Big\},
\end{aligned}$$

where $i = 1, 2, \ldots$.

Proof. Consider the problem (2.1), (2.2). We introduce the following notation

$$x' = y,$$

then the problem (2.1), (2.2) reduces to the Fučik problem for the second order equation

$$y'' = -\mu^2 y^+ + \lambda^2 y^-, \ \mu, \lambda \geq 0, \tag{2.3}$$

$$y^+ = \max\{y, 0\}, \quad y^- = \max\{-y, 0\},$$

$$y(0) = 0 = y(1) \tag{2.4}$$

and one integration of $y(t)$

$$x(t) = x(0) + \int_0^t x'(s)ds = \int_0^t y(s)ds.$$

Notice that the problem (2.3), (2.6) is the classical Fučik problem, which was investigated in the work [1]. Hence the spectrum of the problem (2.1), (2.2) is the same as that for the second order Dirichlet problem that means that it consists of the branches described by the formulae in the theorem formulation. □

Some first branches of the spectrum to the problem (2.1), (2.2) are depicted in Fig. 1.

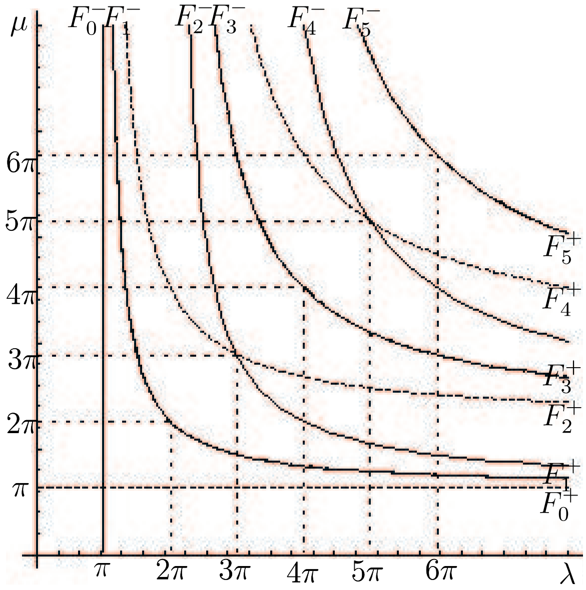


Figure 1: The Fučik spectrum for the problem (2.1), (2.2).

Remark 2.1 *If $\lambda = \mu$ we obtain the eigenvalue problem*

$$x''' = -\lambda^2 x', \quad \lambda \geq 0, \tag{2.5}$$

$$x(0) = x'(0) = 0 = x'(1). \tag{2.6}$$

It follows from the proof of Theorem 2.1 that the eigenvalues of the problem (2.5), (2.6) *are the same as the ones for the problem* (2.1), (2.2). *They are $\lambda_n = \pi n$, where $n = 1, 2, \ldots$.*

Remark 2.2 *Consider the fourth order boundary value problem*

$$x^{(4)} = -\mu^2 x''^{+} + \lambda^2 x''^{-}, \quad \mu, \lambda \geq 0, \tag{2.7}$$

$$x''^{+} = \max\{x'', 0\}, \quad x''^{-} = \max\{-x'', 0\},$$

$$x(0) = x'(0) = x''(0) = 0 = x''(1). \tag{2.8}$$

The Fučik spectrum for this problem is a set of the points (λ, μ) such that the problem has a nontrivial solution.

The Fučik spectrum for this problem is the same as that for the problem (2.1), (2.2).

3 The third order problem with the boundary conditions $\mathbf{x(0) = x'(0) = 0 = x(1)}$

Consider the equation

$$x''' = -\mu^2 x'^{+} + \lambda^2 x'^{-}, \quad \mu, \lambda \geq 0, \tag{3.1}$$

$$x'^{+} = \max\{x', 0\}, \quad x'^{-} = \max\{-x', 0\},$$

with the boundary conditions

$$x(0) = x'(0) = 0 = x(1). \tag{3.2}$$

Decomposition of the Fučik spectrum for the problem (3.1), (3.2) into branches F_i^+ and F_i^- $(i = 1, 2, \ldots)$ is the same as that for the problem (2.1), (2.2).

The next theorem is the main result of this work.

Theorem 3.1 *The Fučik spectrum for the problem* (3.1), (3.2) *consists of the branches given by*

$$\begin{aligned}
F_{2i-1}^+ &= \Big\{(\lambda, \mu) \Big| \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda} - \frac{\mu\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} < 1, \ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},
\end{aligned}$$

$$\begin{aligned}
F_{2i}^+ &= \Big\{(\lambda, \mu) \Big| \frac{(2i+1)\lambda}{\mu} - \frac{2i\mu}{\lambda} - \frac{\lambda\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < 1, \ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},
\end{aligned}$$

$$\begin{aligned}
F_{2i-1}^- &= \Big\{(\lambda, \mu) \Big| \frac{2i\mu}{\lambda} - \frac{(2i-1)\lambda}{\mu} - \frac{\lambda\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0, \\
&\quad \frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} < 1, \ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},
\end{aligned}$$

$$\begin{aligned}
F_{2i}^- &= \Big\{(\lambda, \mu) \Big| \frac{(2i+1)\mu}{\lambda} - \frac{2i\lambda}{\mu} - \frac{\mu\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < 1, \ \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > 1\Big\},
\end{aligned}$$

where $i = 1, 2, \ldots$.

Proof. Consider the problem (3.1), (3.2). We will prove the theorem for the case of F_1^+. Suppose that $(\lambda, \mu) \in F_1^+$ and let $x(t)$ be a respective nontrivial solution of the problem (3.1), (3.2). The derivative of this solution has only one zero in $(0, 1)$. Let this zero be denoted by τ.

Consider a solution of (3.1), (3.2) in the interval $[0, \tau]$. We obtain that the problem (3.1), (3.2) in this interval reduces to the linear eigenvalue problem

$$x''' = -\mu^2 x', \tag{3.3}$$

$$x(0) = x'(0) = 0 = x(\tau). \tag{3.4}$$

Since $x'(t) = C \sin \mu t$ $(C > 0)$ and $x'(\tau) = 0$ we obtain $\tau = \frac{\pi}{\mu}$. It is easy to see that $x(t) = \frac{C}{\mu}(1 - \cos \mu t)$ in $[0, \tau]$. We have also

$$x(\frac{\pi}{\mu}) = \frac{2C}{\mu}, \tag{3.5}$$

$$x''(\frac{\pi}{\mu}) = -\mu C. \tag{3.6}$$

Now we consider a solution of (3.1), (3.2) in $[\tau, 1]$. We have in this interval linear eigenvalue problem

$$x''' = \lambda^2 x', \tag{3.7}$$

$$x(\tau) = 0 = x(1). \tag{3.8}$$

Since $x'(t) = -A \sin(\lambda t - \lambda \frac{\pi}{\mu})$ $(A > 0)$ and in view of (3.5) we obtain $x(t) = \frac{2C}{\mu} - \frac{A}{\lambda} + \frac{A}{\lambda} \cos(\lambda t - \lambda \frac{\pi}{\mu})$ in $[\tau, 1]$. We have also that

$$x''(\frac{\pi}{\mu}) = \lambda A. \tag{3.9}$$

It follows from (3.6) and (3.9) that $C = \frac{\lambda A}{\mu}$.
It follows from the last equality that

$$x(1) = \frac{2\lambda A}{\mu^2} - \frac{A}{\lambda} + \frac{A \cos(\lambda - \frac{\lambda \pi}{\mu})}{\lambda} = 0. \tag{3.10}$$

Dividing (3.10) by A and multiplying by μ, we obtain

$$\frac{2\lambda}{\mu} - \frac{\mu}{\lambda} + \frac{\mu \cos(\lambda - \frac{\lambda \pi}{\mu})}{\lambda} = 0. \tag{3.11}$$

Considering the derivative of the solution of the (3.1), (3.2) it is easy to prove that

$$0 < \frac{\pi}{\mu} < 1 < \frac{\pi}{\mu} + \frac{\pi}{\lambda}.$$

The last result and (3.11) prove the theorem for the case of F_1^+. The proof for other branches is analogous. □

Visualization of the spectrum to the problem (3.1), (3.2) is given in Figure 2.

Remark 3.1 *If $\lambda = \mu$ we obtain the eigenvalue problem*

$$x''' = -\lambda^2 x', \quad \lambda \geq 0, \tag{3.12}$$

$$x(0) = x'(0) = 0 = x(1). \tag{3.13}$$

Easy computation shows that the eigenvalues of the problem (3.12), (3.13) *are given by $\lambda_n = 2\pi n$, where $n = 1, 2, \ldots$.*

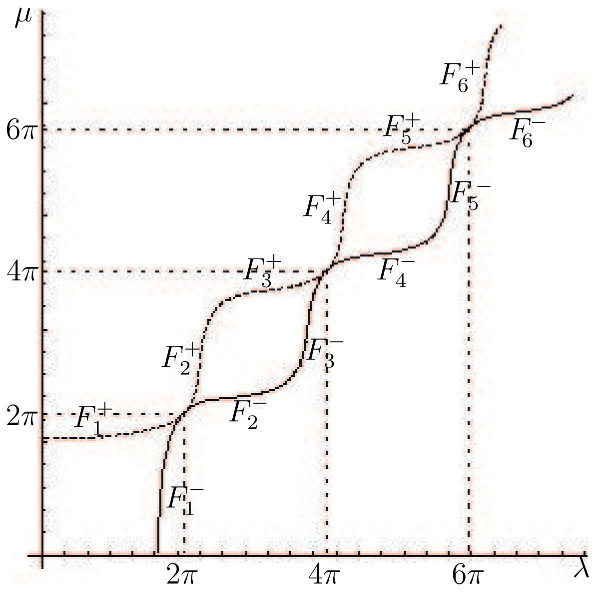


Figure 2: The Fučik spectrum for the problem (3.1), (3.2).

Remark 3.2 *The Fučik spectrum for the fourth order boundary value problem*

$$x'''' = -\mu^2 x''^{+} + \lambda^2 x''^{-}, \quad \mu, \lambda \geq 0, \tag{3.14}$$

$$x''^{+} = \max\{x'', 0\}, \quad x''^{-} = \max\{-x'', 0\},$$

$$x(0) = x'(0) = x''(0) = 0 = x'(1) \tag{3.15}$$

is the same as that for the problem (3.1), (3.2)*.*

Figure 3 shows some solutions which relate to points (λ, μ) on different branches of the spectrum.

4 Connection between the spectra

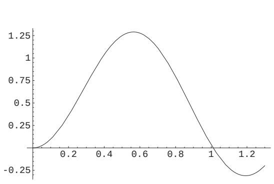
Consider the third order equation

$$x''' = -\mu^2 x'^{+} + \lambda^2 x'^{-}, \quad \mu, \lambda \geq 0, \tag{4.1}$$

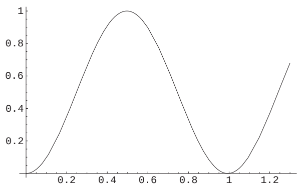
where $x'^{+} = \max\{x', 0\}$, $x'^{-} = \max\{-x', 0\}$ together with the boundary conditions

$$x(0) = x'(0) = 0; \tag{4.2}$$

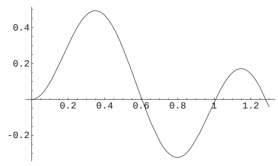
$$\alpha x(1) + (1 - \alpha) x'(1) = 0, \quad \alpha \in [0, 1]. \tag{4.3}$$



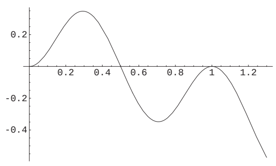
a: $(\lambda, \mu) \in F_1^+$



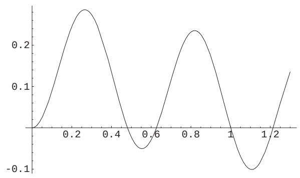
b: $(\lambda, \mu) \in F_1^+ \cap F_2^+$



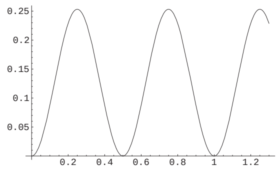
c: $(\lambda, \mu) \in F_2^+$



d: $(\lambda, \mu) \in F_2^+ \cap F_3^+$



e: $(\lambda, \mu) \in F_3^+$



f: $(\lambda, \mu) \in F_3^+ \cap F_4^+$

Figure 3: Some solutions of the problem (3.1), (3.2)

Theorem 4.1 *The Fučik spectrum for the problem* (4.1) – (4.3) *consists of the branches given by*

$$
\begin{aligned}
F_{2i-1}^+ &= \Big\{(\lambda,\mu)\Big|\ \frac{2i\lambda}{\mu}\alpha - \frac{(2i-1)\mu}{\lambda}\alpha - \\
&\quad -\frac{\mu(\alpha\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i) - \lambda\sin(\lambda - \frac{\lambda\pi i}{\mu} + \pi i) + \alpha\lambda\sin(\lambda - \frac{\lambda\pi i}{\mu} + \pi i))}{\lambda} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} < 1,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\}, \\
F_{2i}^+ &= \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\lambda}{\mu}\alpha - \frac{2i\mu}{\lambda}\alpha; \\
&\quad -\frac{\lambda(\alpha\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i) - \mu\sin(\mu - \frac{\mu\pi i}{\lambda} + \pi i) + \alpha\mu\sin(\mu - \frac{\mu\pi i}{\lambda} + \pi i))}{\mu} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < 1,\ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\}, \\
F_{2i-1}^- &= \Big\{(\lambda,\mu)\Big|\ \frac{2i\mu}{\lambda}\alpha - \frac{(2i-1)\lambda}{\mu}\alpha - \\
&\quad -\frac{\lambda(\alpha\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i) - \mu\sin(\mu - \frac{\mu\pi i}{\lambda} + \pi i) + \alpha\mu\sin(\mu - \frac{\mu\pi i}{\lambda} + \pi i))}{\mu} = 0, \\
&\quad \frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} < 1,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\}, \\
F_{2i}^- &= \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\mu}{\lambda}\alpha - \frac{2i\lambda}{\mu}\alpha - \\
&\quad -\frac{\mu(\alpha\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i) - \lambda\sin(\lambda - \frac{\lambda\pi i}{\mu} + \pi i) + \alpha\lambda\sin(\lambda - \frac{\lambda\pi i}{\mu} + \pi i))}{\lambda} = 0, \\
&\quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < 1,\ \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > 1\Big\},
\end{aligned}
$$

where $i = 1, 2, \ldots$.

Proof. The proof of the theorem is analogous to that of the theorem 3.1. □

Remark 4.1 *If* $\alpha = 0$ *we obtain the problem* (2.1), (2.2). *In case of* $\alpha = 1$ *we have the problem* (3.1), (3.2).

The branches $F_1^{\pm}$ to $F_5^{\pm}$ of the spectrum for the problem (4.1) – (4.3) for several values of α are depicted in Figure 4.

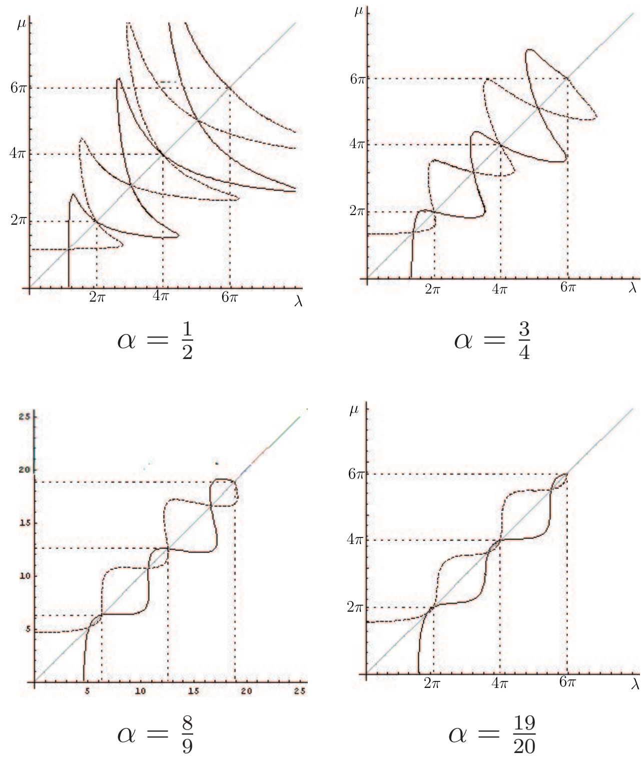


Figure 4: The Fučik spectrum for the problem (4.1) - (4.3) for some values of α

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Н. Сергеева. Спектры Фучика для уравнений третьего и четвертого порядка.

Аннотация. Нами построены спектры Фучика для некоторых специфических дифференциальных уравнений третьего и четвертого порядков. Наши спектры отличаются от известных.

УДК 517.51 + 517.91

N.Sergejeva. Fučika spektri trešās un ceturtās kārtas vienādojumiem.

Anotācija. Mēs konstruējam Fučika spektrus dažiem specifiskiem trešās un ceturtās kārtas diferenciālvienādojumiem. Mūsu spektri atšķirās no zināmiem.

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