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Section: NATURAL SCIENCES, Mathematics and Computer Science

Subsection: **Boundary Value Problems for Ordinary Differential Equations**

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Programme:

- Yu. Klokov. On some boundary value problem arising in the calculus of variations
- A. Lepin, L. Lepin, V. Ponomarev. Boundary value problems for the 3rd order differential equations (*abstract in Russian follows*)
- L. Lepin. Boundary value problems with maximal solution under the 1 - D conditions
- N. Vasilyev, A. Lepin. On some singular boundary value problem
- V. Ponomarev. About uniqueness of a solution of boundary value problems for a system of two first order differential equations with linear boundary conditions (*abstract in Russian follows*)
- A. Gritsans, F. Sadyrbaev. On properties of solutions of the Emden - Fowler type differential equations (*abstract follows*).
- I. Yermachenko. Types of solutions and nonlinear boundary value problems (*abstract follows*).
- S. Atslega. Multiple solutions of nonlinear boundary value problems for ODE
- N. Sergejeva. On the third order boundary value problems (*abstract follows*).
- T. Garbuza. On the sixth order linear differential equations
- A. Gritsans, F. Sadyrbaev. Multiplicity of the Nehari solutions (*abstract follows*).

Некоторые краевые задачи для дифференциального уравнения третьего порядка

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Для краевой задачи

$$(\varphi(t, x, x''))' = Fx, \quad t \in I = [a, b], \tag{1}$$

$$\varphi(a, x(a), x''(a)) = A, \quad H_1 x = h_1, \quad H_2 x = h_2, \tag{2}$$

$$\alpha \leq x \leq \beta, \tag{3}$$

где $\varphi \in C(I \times R^2, R)$ не убывает по x и строго возрастает по x'', $F \in C(C^1(I, R), L_1(I, R))$, $A \in R$, $H_1, H_2 \in C(C^1(I, R), R)$, $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$, $h_1 \in [H_1\alpha, H_1\beta]$, $h_2 \in [H_2\alpha, H_2\beta]$, $\alpha \in C^1(I, R)$ – нижняя функция и $\beta \in C^1(I, R)$ – верхняя функция, которые удовлетворяют следующим условиям: $\alpha \leq \beta$, $\alpha', \beta' \in Lip(I, R)$, для любых $r \in (a, b)$ и $s \in (r, b)$ из существования производных $\alpha''(r)$, $\alpha''(s)$, $\beta''(r)$ и $\beta''(s)$ следуют неравенства

$$\varphi(s, \alpha(s), \alpha''(s)) - \varphi(r, \alpha(r), \alpha''(r)) \geq \int_r^s F\alpha du,$$

$$\varphi(s, \beta(s), \beta''(s)) - \varphi(r, \beta(r), \beta''(r)) \leq \int_r^s F\beta du,$$

$$\varphi(a, \alpha(a), \alpha''(a)) \geq A, \quad \varphi(a, \beta(a), \beta''(a)) \leq A,$$

для любого $x \in C^1(I, R)$ из $\alpha \leq x \leq \beta$ следует

$$F\beta \leq Fx \leq F\alpha,$$

найдены условия на H_1 и H_2, которые обеспечивают разрешимость краевой задачи (1)-(3).

Аналогичная краевая задача рассматривалась в работе [1].

References

[1] A.Cabada, S.Heikkila, Existence of solutions of third-order functional problems with non-linear boundary conditions. ANZIAM J. (2004) 46, 1, 33-44.

About uniqueness of a solution of boundary value problems for a system of two first order differential equations with linear boundary conditions

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Consider the system of two differential equations

$$x' = h(t, x, y), \quad y' = f(t, x, y) \tag{1}$$

together with the following boundary conditions

$$\begin{aligned} a_1 x(a) + a_2 x(b) + a_3 y(a) + a_4 y(b) + a_5 = 0, \\ b_1 x(a) + b_2 x(b) + b_3 y(a) + b_4 y(b) + b_5 = 0, \end{aligned} \tag{2}$$

where $h, f \in Car([a,b] \times R^2, R)$, $-\infty < a < b < +\infty$, $a_i, b_i \in R$, $i = 1, ..., 5$, $\Delta_{ij} = a_i b_j - a_j b_i$, $i, j \in \{1, ..., \}$.

We prove the following result.

Theorem. Suppose that $h(t,x,y)$ is increasing in y and $f(t,x,y)$ is strictly increasing in x. Let also the conditions

$$h(t, x_1, y_1) - h(t, x_2, y_1) \le K(t)(x_1 - x_2), \quad x_1 \le x_2;$$

$$h(t, x_1, y_1) - h(t, x_2, y_1) \ge K_1(t)(x_1 - x_2), \quad x_1 \ge x_2,$$

$$f(t, x_1, y_1) - f(t, x_1, y_2) \le K_2(t)(y_1 - y_2), \quad y_1 \ge y_2,$$

$$f(t, x_1, y_1) - f(t, x_1, y_2) \ge K_3(t)(y_1 - y_2), \quad y_1 \ge y_2,$$

$$f(t, x_1, y_1) - f(t, x_1, y_2) \ge K_4(t)(y_1 - y_2), \quad y_1 \le y_2,$$

$\Delta_{23} \ne 0$, $\varepsilon\Delta_{12} \ge 0$, $\varepsilon\Delta_{13} \ge 0$, $\varepsilon\Delta_{24} \ge 0$, $\varepsilon\Delta_{43}$ hold, $K_i \in L([a,b], (0, +\infty))$, $i = 1, ..., 4$, $\varepsilon = \operatorname{sign} \Delta_{23}$. Then the boundary value problem (1), (2) has at most one solution.

Some properties of solutions of Emden-Fowler type equations

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In the study of various classes of nonlinear differential equations it is useful sometimes to clarify properties of solutions of similar autonomous equations. For example, an important role play the generalized sine and cosine functions in investigations of half-linear differential equations [1]. Similarly, the lemniscatic functions are applied in the Nehari variational theory [2].

Denote by S and C solutions of the Emden - Fowler type differential equation $x'' = -3x^5$, which satisfy respectively the initial conditions $x(0) = 0,\ x'(0) = 1$ un $x(0) = 1,\ x'(0) = 0$. We mention the following properties of S and C.

- The functions S and C are periodic in the entire R with the minimal period $T = 4\int_0^1 \frac{dt}{\sqrt{1-t^6}}$ and the value range is $[-1;1]$.
- The functions S and C can be expressed in terms of the Jacobian elliptic functions:
$$S(t) = \frac{(\alpha t;k)}{\sqrt{\alpha^{22}\,(\alpha t;k) +^2 (\alpha t;k)}},\ C(t) = \frac{(\alpha t;k)}{\sqrt{^2\,(\alpha t;k) + \alpha^{22}\,(\alpha t;k)}},$$
$$\alpha = \sqrt[4]{3},\quad k = \frac{\sqrt{2-\sqrt{3}}}{2},\quad \alpha^2 = 2 - 4k^2.$$
- for any real t the following analogue of the basic trigonometric identity is valid
$$S^2(t) + 2S^2(t)C^2(t) + C^2(t) = 1 \ \text{ or } \ \big(1 + 2S^2(t)\big)\big(1 + 2C^2(t)\big) = 3.$$
- for any real t
$$S'(t) = C(t)\sqrt{1 + 3S^2(t) + 3S^4(t) + 2S^6(t)},$$
$$C'(t) = -S(t)\sqrt{1 + 3C^2(t) + 3C^4(t) + 2C^6(t)}.$$
- the functions S and C satisfy the addition theorem, that is, there exist two argument functions Φ and Ψ such that for any real u and v the relations
$$S(u+v) = \Phi\big(S(u); S(v)\big) \ \text{ and } \ C(u+v) = \Psi\big(C(u); C(v)\big)$$
hold.

References

[1] O. Došlý, J. Řezníčková, *Regular half-linear second order differential equation,* Archivum Mathematicum (Brno), **39**, 2003, 233–245.

[2] A. Gritsans and F. Sadyrbaev, *Characteristic numbers of non-autonomous Emden-Fowler type euations*, in: R. Čiegis, edr., Proceedings of the 10th International Conference MMA2005&CMAM2, June 1-5, 2005, Trakai, Lithuania, pp. 403–408.

On solutions of the fourth-order nonlinear boundary value problem

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We consider the fourth-order nonlinear problem $x^{(4)} = q(t) \cdot |x|^p \operatorname{sgn} x$ (i), $x(0) = x'(0) = 0 = x(1) = x'(1)$ (ii), where $p > 0,\ p \neq 1,\ t \in I := [0,1],\ q(t) \in C(I, (0,\ +\infty))$.

We investigate the problem (i), (ii) using the quasilinearization process [1]. Namely, we represent the equation (i) in a quasilinear form $x^{(4)} - k^4 x = F(t,\, x)$ (iii) with non-resonant linear part in order the equations (i) and (iii) be equivalent in some domain. We proved in [3] that modified problem (iii), (ii) has a solution of the same oscillatory type as the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ has. We showed in [2] that under certain conditions this quasilinearization process can be applied with essentially different linear parts and hence the original problem is shown to have multiple solutions.

Theorem 0.1 *Suppose that* $0 < q_1 \leq q(t) \leq q_2$ $\forall t \in [0,\ 1]$. *If there exists some* k *in the form* $k = \pi i$, $(i = 1,\, 2, \ldots)$, *which satisfies the inequality*

$$k \cdot \frac{e^k(4\sqrt{2}+3)-1}{4(e^k+1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}\Big(\frac{q_1}{q_2}\Big)^{\frac{1}{|p-1|}} \qquad \textit{for} \quad k = (2n-1)\pi$$

or

$$k \cdot \frac{e^k(4\sqrt{2}+3)+1}{4(e^k-1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}\Big(\frac{q_1}{q_2}\Big)^{\frac{1}{|p-1|}} \qquad \textit{for} \quad k = 2n\pi,$$

where β *is a positive root of the equation* $\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}$, *then there exists an* $(i-1)$*-type solution of the problem (i), (ii).*

References

[1] I. Yermachenko, F. Sadyrbaev. Types of solutions and multiplicity results for two-point nonlinear boundary value problems. - Elsevier Ltd., Nonlinear Analysis **63** (2005), pp. e1725-e1735. (electr.version http://www.elsevier.com/locate/na)

[2] I.Yermachenko. Multiple Solutions of the Fourth-Order Emden-Fowler Equation. - In: R. Čiegis (Ed.), Proc. of the 10-th Int.conf. MMA2005&CMAM2, Trakai, Lithuania, (2005), pp. 547–552.

[3] F. Sadyrbaev and I. Yermachenko. Types of solutions and multiplicity results for the fourth-order boundary value problems.- Proc. of the Conference on Differential and Difference Equations, Melbourne, FL, USA, August 2005, (to appear).

On Fučik spectra for the third and fourth order equations

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We construct the Fučik spectrum for the third order nonlinear boundary value problem

$$x''' = -\mu^2 x'^{+} + \lambda^2 x'^{-}, \ \mu, \lambda > 0, \quad x'^{\pm} = \max\{\pm x', 0\}, \tag{1}$$

$$x(0) = x'(0) = 0 = x(1). \tag{2}$$

By the Fučik spectrum we mean the set

$$\{(\lambda, \mu) : \text{ the problem (1), (2) has a nontrivial solution}\}.$$

Theorem. *The Fučik spectrum for the problem (1), (2) consists of the branches given by*

$$\begin{aligned}
F^{+}_{2n-1} &= \Big\{(\lambda, \mu)\Big| \ \tfrac{2n\lambda}{\mu} - \tfrac{(2n-1)\mu}{\lambda} - \tfrac{\mu\cos(\lambda - \frac{\lambda\pi n}{\mu} + \pi n)}{\lambda} = 0, \\
&\quad \tfrac{n\pi}{\mu} + \tfrac{(n-1)\pi}{\lambda} < 1, \ \tfrac{n\pi}{\mu} + \tfrac{n\pi}{\lambda} > 1\Big\}, \\
F^{+}_{2n} &= \Big\{(\lambda, \mu)\Big| \ \tfrac{(2n+1)\lambda}{\mu} - \tfrac{2n\mu}{\lambda} - \tfrac{\lambda\cos(\mu - \frac{\mu\pi n}{\lambda} + \pi n)}{\mu} = 0, \\
&\quad \tfrac{n\pi}{\mu} + \tfrac{n\pi}{\lambda} < 1, \ \tfrac{(n+1)\pi}{\mu} + \tfrac{n\pi}{\lambda} > 1\Big\}, \\
F^{-}_{2n-1} &= \Big\{(\lambda, \mu)\Big| \ \tfrac{2n\mu}{\lambda} - \tfrac{(2n-1)\lambda}{\mu} - \tfrac{\lambda\cos(\mu - \frac{\mu\pi n}{\lambda} + \pi n)}{\mu} = 0, \\
&\quad \tfrac{(n-1)\pi}{\mu} + \tfrac{n\pi}{\lambda} < 1, \ \tfrac{n\pi}{\mu} + \tfrac{n\pi}{\lambda} > 1\Big\}, \\
F^{-}_{2n} &= \Big\{(\lambda, \mu)\Big| \ \tfrac{(2n+1)\mu}{\lambda} - \tfrac{2n\lambda}{\mu} - \tfrac{\mu\cos(\lambda - \frac{\lambda\pi n}{\mu} + \pi n)}{\lambda} = 0, \\
&\quad \tfrac{n\pi}{\mu} + \tfrac{n\pi}{\lambda} < 1, \ \tfrac{n\pi}{\mu} + \tfrac{(n+1)\pi}{\lambda} > 1\Big\},
\end{aligned}$$

where $n = 1, 2, \ldots$.

We investigate also properties of solutions to the problem (1), (2) and made comparison with some known Fučik type problems. Let us mention that the Fučik spectrum for the problem under consideration significantly differs from known Fučik spectra (cfn. [1], [2]).

We consider also some fourth order problem and investigate its Fučik spectrum. It is similar to that for the problem (1), (2).

References

[1] A. Kufner and S. Fučik. Nonlinear Differential Equations. Nauka, Moscow, 1988(in Russian).

[2] Gaudenzi M. and Habets P. Fučik Spectrum for a Third Order Equation. J. Dif. Equations, 1996, 128, 556-595.

Multiplicity of the Nehari solutions

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The boundary value problem (BVP)

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0, \quad q \in C(R,(0,+\infty)), \tag{1}$$

$$x(a) = 0 = x(b), \quad x(t) \text{ has } n-1 \text{ zeros in } (a,b) \tag{2}$$

for a fixed integer n may have multiple solutions $x_n(t)$. The Nehari solutions are those solutions of the BVP (1), (2) which minimize the functional

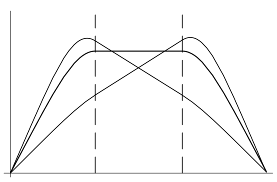
$$J(x) := \frac{\varepsilon}{1+\varepsilon}\int_a^b q(t)x_n^{2+2\varepsilon}(t)\,dt \left(= \min \frac{\varepsilon}{1+\varepsilon}\int_a^b x_n'^2(t)\,dt\right).$$

It was proven in [1], that there exists at least one Nehari solution, so the problem is always solvable. We construct the examples showing that there may be multiple Nehari solutions. In the first example equation is of the form

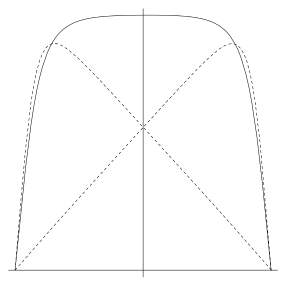
$$x'' = -q(t)x^3, \quad q(t) \geq 0, \tag{3}$$

where $q(t)$ is “a constant - zero - constant"function, defined on the interval $[0,1] \cup [1,1+T] \cup [1+T,2+T]$. In the second example equation is also of the form (3), where $q(t) = \dfrac{2}{\left(\xi(t)\right)^6}$ and $\xi(t)$ is a Λ-shaped function in the interval $[-1,1]$.

In both cases for appropriate choice of parameters there exist exactly three solutions of the BVP and the two asymmetrical ones are the Nehari solutions.



Example 1.



Example 2.

References

[1] Z. Nehari. Characteristic values associated with a class of nonlinear second-order differential equations. *Acta Math.*, 105: 141 – 175, 1961.