Fuchik spectrum for the second order Sturm-Liouville boundary value problem [1]

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Summary. We provide the explicit formulas for the Fuchik spectrum of the boundary value problem

$$x'' = -\mu^2 x^+ + \lambda^2 x^-,$$

$$\mu,\ \lambda \in \mathbb{R}, \quad x^{\pm}(t) = \max\{\pm x, 0\},$$

$$\begin{cases} x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0. \end{cases}$$

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Introduction

We consider the equation with the piece-wise linear right side

$$x'' = -\mu^2 x^+ + \lambda^2 x^-, \quad \mu,\ \lambda \in \mathbb{R}, \tag{1}$$

where $x^{\pm}(t) = \max\{\pm x, 0\}$, with the Sturm-Liouville boundary conditions

$$\begin{cases} x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0, \end{cases} \tag{2}$$

where $0 \leq \alpha \leq \dfrac{\pi}{2} \leq \beta \leq \pi$. We are looking for those values of (λ, μ) for which the problem has a nontrivial solution.

Definition. By **the Fuchik spectrum** for the problem (1), (2) is called the set of all values (λ, μ) such that a nontrivial solution for the problem (1), (2) exists.

In this paper we show that the spectrum of the problem (1), (2) is a collection of curves and we obtain the formulas for the spectrum. We show that the branches of the spectrum are hyperbolas or straight lines, which are denoted by F_n^+ and F_n^-, where the

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lower index indicates how many zeros in the interval $(0,\pi)$ has the respective solution $x(t)$ of (1), (2) and the upper index $(+)$ shows that $x'(0)>0$, respectively $(-)$ shows that $x'(0)<0$.

The paper is organized as follows. Section 1 provides auxiliary results and describes the technique we used in the sequel. Section 2 is devoted to the main results. In Section 3 we discuss the specific cases, namely, the Neumann boundary conditions, the mixed boundary conditions etc.

All the other cases of ordering of α and β are considered in the author's Master work (see [2]).

1 Auxiliary results

Our technique is based on a regular usage of polar coordinates. Let us introduce them by the formulas $x=\rho\sin\varphi$ and $x'=\rho\cos\varphi$.

The piece-wise linear function $f(x)=-\mu^2x^+ + \lambda^2x^-$, in polar coordinates looks as

$$f(\rho,\varphi)=\begin{cases} -\mu^2\rho\sin\varphi, & \sin\varphi\ge 0,\\ -\lambda^2\rho\sin\varphi, & \sin\varphi<0.\end{cases} \tag{3}$$

Proposition 1.1 *Let $\varphi(t)$ be the angle function for solutions of the Cauchy problem* (1), *$\varphi(0)=\varphi_0,\ \rho(0)=\rho_0$. The difference $\varphi(T)-\varphi(0)$ is independent of the choice of $\rho_0>0$, that is, any trajectory starting at the time moment $t=0$ from the first of the straight lines (2) on a phase plane, ends at some other straight line $x(T)\cos\varphi(T)-x'(T)\sin\varphi(T)=0$.*

Proof. We differentiate $x=\rho\sin\varphi$ and $x'=\rho\cos\varphi$ in t, and obtain

$$\begin{aligned} x' &= \rho'\sin\varphi+\rho\varphi'\cos\varphi,\\ x'' &= \rho'\cos\varphi-\rho\varphi'\sin\varphi.\end{aligned}$$

One obtains from (1) and $x'=\rho\cos\varphi$ the system of two equations:

$$\begin{cases} \rho'\sin\varphi+\rho\varphi'\cos\varphi=\rho\cos\varphi,\\ \rho'\cos\varphi-\rho\varphi'\sin\varphi=f(\rho,\varphi).\end{cases}$$

Finally one has the system:

$$\begin{cases} \rho'=\rho\sin\varphi\cos\varphi+f(\rho,\varphi)\cos\varphi,\\ \varphi'=-\dfrac{f(\rho,\varphi)}{\rho}\sin\varphi+\cos^2\varphi.\end{cases}$$

Let us rewrite the second equation in the form

$$\varphi'=F(\varphi)=\begin{cases} \mu^2\sin^2\varphi+\cos^2\varphi, & \sin\varphi\ge 0,\\ \lambda^2\sin^2\varphi+\cos^2\varphi, & \sin\varphi<0.\end{cases} \tag{4}$$

As can be seen from (4), the derivative of $\varphi(t)$ is independent of $\rho(t)$. Notice that the function $\varphi(t)$ is increasing, since $\varphi'>0$.

Proposition 1.2 *A solution of the problem*

$$\begin{cases} \varphi' = k^2 \sin^2 \varphi + \cos^2 \varphi, \ k > 0, \\ \varphi(t_0) = \varphi_0 \end{cases}$$

is given by

$$\frac{1}{k} \arctan(k \tan \varphi) - \frac{1}{k} \arctan(k \tan \varphi_0) = t - t_0,$$

for $0 \le \varphi_0 \le \varphi \le \dfrac{\pi}{2}$ *or* $\dfrac{\pi}{2} \le \varphi_0 \le \varphi \le \pi$, *and it is given by*

$$\frac{1}{k} \arctan(k \tan \varphi) + \frac{\pi}{k} - \frac{1}{k} \arctan(k \tan \varphi_0) = t - t_0,$$

for $0 \le \varphi_0 < \dfrac{\pi}{2} < \varphi \le \pi$.

Proof. Consider

$$\frac{d\varphi}{dt} = k^2 \sin^2 \varphi + \cos^2 \varphi, \qquad \frac{d\varphi}{k^2 \sin^2 \varphi + \cos^2 \varphi} = dt.$$

One has, after integration, that

$$\int\limits_{\varphi_0}^{\varphi} \frac{d\varphi}{k^2 \sin^2 \varphi + \cos^2 \varphi} = \int\limits_{t_0}^{t} dt,$$

or

$$t - t_0 = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi}{k^2 \sin^2 \varphi + \cos^2 \varphi} = \int\limits_{\varphi_0}^{\varphi} \frac{\frac{d\varphi}{\cos^2 \varphi}}{k^2 \tan^2 \varphi + 1} = \frac{1}{k} \int\limits_{\varphi_0}^{\varphi} \frac{d(k \tan \varphi)}{(k \tan \varphi)^2 + 1} =$$
$$= \frac{1}{k} \arctan(k \tan \varphi) \Big|_{\varphi_0}^{\varphi} = \frac{1}{k} \arctan(k \tan \varphi) - \frac{1}{k} \arctan(k \tan \varphi_0).$$

Consider the case when there is a value $\varphi_* = \dfrac{\pi}{2}$ in the interval $(\varphi_0; \varphi)$. Then the integral in the right side

$$\int\limits_{t_0}^{t} dt = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi}{k^2 \sin^2 \varphi + \cos^2 \varphi}$$

reduces to the sum of two improper integrals:

$$t - t_0 = \frac{1}{k}\int\limits_{\varphi_0}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2+1} = \frac{1}{k}\left(\int\limits_{\varphi_0}^{\frac{\pi}{2}} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2+1} + \int\limits_{\frac{\pi}{2}}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2+1}\right) =$$

$$= \frac{1}{k}\left(\lim_{\varepsilon_1\to 0}\int\limits_{\varphi_0}^{\frac{\pi}{2}-\varepsilon_1} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2+1} + \lim_{\varepsilon_2\to 0}\int\limits_{\frac{\pi}{2}+\varepsilon_2}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2+1}\right) =$$

$$= \frac{1}{k}\Bigg(\lim_{\varepsilon_1\to 0}\Big[\arctan(k\tan(\frac{\pi}{2}-\varepsilon_1)) - \arctan(k\tan\varphi_0)\Big] +$$

$$+ \lim_{\varepsilon_2\to 0}\Big[\arctan(k\tan\varphi) - \arctan(k\tan(\frac{\pi}{2}+\varepsilon_2))\Big]\Bigg) =$$

$$= \frac{1}{k}\left(\frac{\pi}{2} - \arctan(k\tan\varphi_0) + \arctan(k\tan\varphi) + \frac{\pi}{2}\right) =$$

$$= \frac{1}{k}\arctan(k\tan\varphi) + \frac{\pi}{k} - \frac{1}{k}\arctan(k\tan\varphi_0).$$

Proposition 1.3 *The equality*

$$\int\limits_{\alpha}^{\beta} \frac{d\varphi}{k^2\sin^2\varphi + \cos^2\varphi} = \int\limits_{\alpha+\pi}^{\beta+\pi} \frac{d\varphi}{k^2\sin^2\varphi + \cos^2\varphi} \quad \textit{holds for any } \alpha \textit{ and } \beta.$$

Proof. The proof follows from periodicity of the function $\sin^2\varphi$ and $\cos^2\varphi$.

Let us interpret the Sturm-Liouville boundary conditions (2) on a phase plane. From (2) we have

$$\frac{x(0)}{x'(0)} = \tan\alpha, \qquad \varphi_0 = \alpha,$$
$$\frac{x(\pi)}{x'(\pi)} = \tan\beta, \qquad \text{and} \qquad \varphi_1 = \beta + \pi n, \ \text{for some } n \in \mathbb{N}.$$

where $\varphi_0 = \varphi(0)$ un $\varphi_1 = \varphi(\pi)$. See Fig. 1.

2 Main results

Czech mathematician S. Fuchik in 70-th of the XX-th century formulated and solved a number of problems which relate to the theory of nonlinear differential equations depending on two parameters. The second order Dirichlet boundary value problem was considered in the book ([1]).

The spectrum for the Fuchik problem consists of separate curves.

We consider now the more general case.

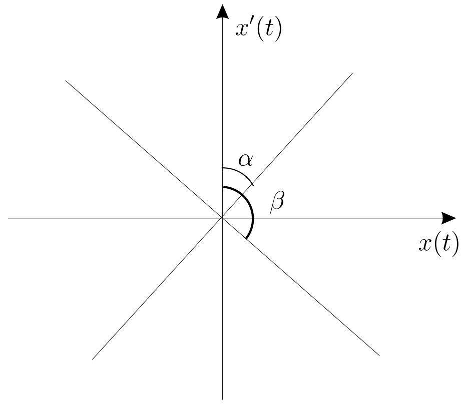

Figure 1: The Sturm-Liouville boundary conditions on a phase plane.

Theorem 2.1 *The spectrum of the problem* (1), (2) *for the case of* $0 \le \alpha \le \frac{\pi}{2} \le \beta \le \pi$ *consists of separate branches*

$$F_0^+ : \frac{1}{\mu}\arctan(\mu\tan\beta) + \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha) = \pi,$$

$$F_0^- : \frac{1}{\lambda}\arctan(\lambda\tan\beta) + \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha) = \pi,$$

$$F_{2k}^+ : \left[\frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha)\right] + \frac{(k-1)\pi}{\mu} + \frac{k\pi}{\lambda} + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi,$$

$$F_{2k}^- : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha)\right] + \frac{k\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi,$$

$$F_{2k-1}^+ : \left[\frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha)\right] + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi,$$

$$F_{2k-1}^- : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha)\right] + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi.$$

$$\forall k \in \mathbb{N}, \quad \lambda > 0, \quad \mu > 0.$$

Proof. One can find spectrum of the problem (1), (2) by resolving the equation

$$\varphi' = \begin{cases} \mu^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi \ge 0, \\ \lambda^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi < 0 \end{cases} \tag{5}$$

together with the boundary conditions $\varphi(0) = \alpha$, $\varphi(\pi) = \beta$.

Consider a solution of the problem (1), (2), which has not zeros in the interval $(0;\pi)$. Then $\varphi_0 = \varphi(0) = \alpha \in [0;\frac{\pi}{2}]$ and $\varphi_1 = \varphi(\pi) = \beta \in [\frac{\pi}{2};\pi]$. This means that for any $t \in (0;\pi)$ a solution $x(t) > 0$. Thus $x(t)$ is a solution of $x'' = -\mu^2 x$ and in polar coordinates satisfy $\varphi'(t) = \mu^2\sin^2\varphi + \cos^2\varphi$, $\varphi(0) = \alpha$, $\varphi(\pi) = \beta$. By Proposition 1.2

$$t - t_0 = \pi = \frac{1}{\mu}\arctan(\mu\tan\beta) + \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha). \tag{6}$$

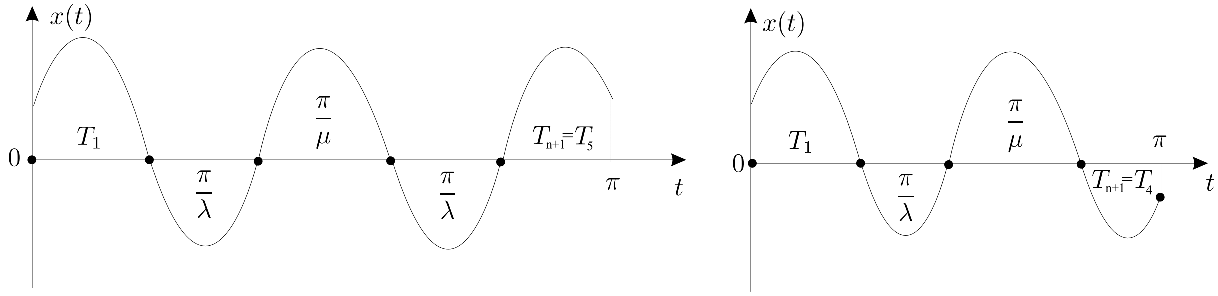

Figure 2: The example of solutions of problem (1), (2) with $n = 4$ and $n = 3$ zeros in the interval $(0;\pi)$.

Since λ is arbitrary positive number, we get the expression for

$$F_0^+ = \{\ (\mu_0, \lambda),\ \text{where } \mu_0 \text{ is a solution of } (6)\}.$$

Similarly, the expression for F_0^- is

$$F_0^- = \{(\lambda_0, \mu),\ \mu \in \mathbb{R}^+\},$$

and λ_0 can be obtained from

$$\pi = \frac{1}{\lambda}\arctan(\lambda\tan\beta) + \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha).$$

When treating the case of $x(t) < 0$ for any $t \in (0;\pi)$, we use also the results of Proposition (1.3) and (1.2).

Next we consider the case of $x(t)$ having exactly one zero, say, at $t = t_1$. Then for $0 \le t \le t_1$ we use $\mu^2 = -\mu^2 x$ and the length of the interval $[0;t_1]$ is, by Proposition (1.2)

$$t_1 - 0 = t_1 = \frac{1}{\mu}\arctan(\mu\tan\pi) + \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha),$$

or

$$t_1 - 0 = t_1 = \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha).$$

For $t_1 \le t \le \pi$, $x(t)$ is negative, therefore we consider a solution of $x'' = -\lambda^2 x$ and by Proposition (1.2) it is equal to

$$\pi - t_1 = \frac{1}{\lambda}\arctan(\lambda\tan(\beta+\pi)) + \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\pi),$$

or by Proposition (1.3)

$$\pi - t_1 = \frac{1}{\lambda}\arctan(\lambda\tan\beta) + \frac{\pi}{\lambda}.$$

The sum of two intervals is π, and we got the expression for $F_1^+ = \{\ (\lambda,\mu)\}$, where λ and μ can be obtained from

$$F_1^+ : \left[\frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha)\right] + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi,$$

Similarly the expression for F_1^- is

$$F_1^- : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha)\right] + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi.$$

For any solution of the problem (1), (2), which has exactly $n > 0$ zeros in the interval $(0;\pi)$, the interval $[0;\pi]$ can be decomposed in $n+1$ subintervals $J_{T_1} := [0;T_1]$, $J_{T_2} := [T_1;T_1+T_2]$, $J_{T_3} := [T_1+T_2;T_1+T_2+T_3]$, ..., $J_{T_{n+1}} := [\sum\limits_{n=1}^{n} T_n;\pi]$ so, that in any of those subintervals the sign of a solution does not change (see Fig. 2), and we use the equations $x'' = -\mu^2 x$ and $x'' = -\lambda^2 x$, depending on the sign of $x(t)$ in the respective interval. We can compute the length of each subinterval making use of the results of Propositions 1.2 and 1.3. The total length of all $n+1$ subintervals is π. This is the basis for obtention of relations between μ and λ.

It is clear, that when finding the analytical description of the branches F_n^+ $(\forall n \in \mathbb{N})$, decomposition of the main interval in subintervals is such that

$$T_1 = \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha),$$

$$T_2 = T_4 = T_6 = \ldots = \frac{\pi}{\lambda},$$

$$T_3 = T_5 = T_7 = \ldots = \frac{\pi}{\mu},$$

$$T_{n+1} = \begin{cases} \frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta), & n \text{ is even}, \\ \frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta), & n \text{ is odd}. \end{cases}$$

In the case of the branches F_n^- $(\forall n \in \mathbb{N})$ this decomposition is such that

$$T_1 = \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha),$$

$$T_2 = T_4 = T_6 = \ldots = \frac{\pi}{\mu},$$

$$T_3 = T_5 = T_7 = \ldots = \frac{\pi}{\lambda},$$

$$T_{n+1} = \begin{cases} \frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta), & n \text{ is even}, \\ \frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta), & n \text{ is odd}. \end{cases}$$

We use the fact, that the sum of the lengths of all intervals J_{T_1}, J_{T_2}, J_{T_3}, ..., $J_{T_{n+1}}$ is π,

$$\sum_{i=1}^{n+1} T_i = \pi,$$

and obtain the Fuchik spectrum for the problem (1), (2).

2.1 Properties of the Fuchik spectrum for the Sturm-Liouville problem

We consider the function

$$\psi(z,\varphi) = \frac{1}{z}[\pi - \arctan(z\tan\varphi)],\ z > 0. \tag{7}$$

Lemma 2.1 *The function $\psi(z,\varphi)$ is a monotonically decreasing function with respect to both z and φ.*

Proof. The proof immediately follows from monotonicity of the functions $\dfrac{1}{z}$ and $\arctan(z)$.

By using the formula (7), we can reduce the formulas for the branches of the Fuchik spectrum to the form

$$F^+_{2k-1}:\ \psi(\mu,\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \psi(\lambda,\gamma) = \pi, \tag{8}$$

$$F^-_{2k-1}:\ \psi(\lambda,\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \psi(\mu,\gamma) = \pi. \tag{9}$$

$$F^+_{2k}:\ \psi(\mu,\alpha) + \frac{(k-1)\pi}{\mu} + \frac{k\pi}{\lambda} + \psi(\mu,\gamma) = \pi, \tag{10}$$

$$F^-_{2k}:\ \psi(\lambda,\alpha) + \frac{k\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \psi(\lambda,\gamma) = \pi, \tag{11}$$

where $\pi - \beta = \gamma \in [0; \frac{\pi}{2}]$.

Proposition 2.1 *The branches F^+_n and F^-_n are symmetrical with respect to the bisectrix of the first quadrant. This means that $\forall n \in \mathbb{N}$, for each $(u,v) \in F^+_n$, there exists $(v,u) \in F^-_n$ and vice versa.*

Proof. The formulas (8) and (9) are symmetrical with respect to replacement of μ by λ and vice versa. The same is true for (10) and (11).

Proposition 2.2 *The branches F^+_n and F^-_n intersect at the point (ν_n,ν_n) where ν_n^2 is the respective simple eigenvalue of the problem*

$$\begin{cases} x'' = -\nu^2 x, \\ x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0, \quad 0 \le \alpha \le \dfrac{\pi}{2} \le \beta \le \pi. \end{cases}$$

Proof. The proof follows from the proposition 2.1.

Proposition 2.3 *If $\alpha = \gamma$, then branches F_{2k-1}^{+} and F_{2k-1}^{-} coincide.*

Proof. If $\alpha = \gamma$, then the formulas (8) and (9) are identical.

Proposition 2.4 *The Fuchik branches $F_n^{\pm}$ are the graphs of monotonically decreasing functions.*

Proof. This can be seen from the relations (8), (10), (9), (11).

Proposition 2.5 *For any branch $F_n^{\pm}$, where $n \in \mathbb{N}$ is fixed, there exist a vertical asymptote $\lambda = \lambda_n^{\pm}$ and a horizontal asymptote $\mu = \mu_n^{\pm}$, $\forall n \in \mathbb{N}$.*

Proof. We discuss the case of the branch F_{2k}^{+}. If $\mu \to +\infty$, then $\psi(\mu, \alpha) \to 0$, $\psi(\mu, \gamma) \to 0$ and $\dfrac{(k-1)\pi}{\mu} \to 0$. It follows from (10) that $\dfrac{k\pi}{\lambda} \to \pi$ as $\mu \to +\infty$. The function $\mu = f(\lambda)$ is monotone, therefore $\lambda \to \lambda_{2k}^{+}$, where $\lambda_{2k}^{+} = k$ is the vertical asymptote.

By repetition of the argument of the previous case, we have that the horizontal asymptote $\mu = \mu_{2k}^{+}$ is a solution of

$$\pi[(1+k) - \mu] = \arctan(\mu \tan \alpha) + \arctan(\mu \tan \gamma).$$

We have that the branches F_{2k}^{+} and F_{2k}^{-} are symmetrical, therefore their vertical and horizontal asymptotes are symmetrical. The asymptotes for F_{2k}^{-}, where k is fixed, are $\mu_{2k}^{-} = k$ and λ_{2k}^{-}, which is a solution of the equation:

$$\pi[(1+k) - \lambda] = \arctan(\lambda \tan \alpha) + \arctan(\lambda \tan \gamma).$$

The proof for the branches $F_{2k-1}^{\pm}$ is similar. The vertical asymptote λ_{2k-1}^{+} is a solution of equation $\pi(k - \lambda) = \arctan(\lambda \tan \gamma)$, $\mu = \mu_{2k-1}^{+}$ is a solution of $\pi(k - \mu) = \arctan(\mu \tan \alpha)$, but μ_{2k-1}^{-} and λ_{2k-1}^{-} are symmetric with λ_{2k-1}^{+} and μ_{2k-1}^{+}.

Theorem 2.2 (Comparison of asymptotes.) *The asymptotes of the branches F_{2k-1}^{+} and F_{2k-1}^{-} relate as shown $\forall k \in \mathbb{N}$:*

If $\alpha > \gamma$, then

$$\lambda_{2k-1}^{-} < \lambda_{2k-1}^{+}, \quad \mu_{2k-1}^{-} > \mu_{2k-1}^{+}.$$

If $\alpha < \gamma$, then

$$\lambda_{2k-1}^{-} > \lambda_{2k-1}^{+}, \quad \mu_{2k-1}^{-} < \mu_{2k-1}^{+}.$$

See Fig. 3.

The asymptotes of the branches F_{2k}^{+} and F_{2k}^{-} relate as shown:

$$\lambda_{2k}^{-} > \lambda_{2k}^{+}, \quad \mu_{2k}^{-} < \mu_{2k}^{+}, \quad \forall \alpha, \gamma \ \forall k \in \mathbb{N}.$$

See Fig. 4.

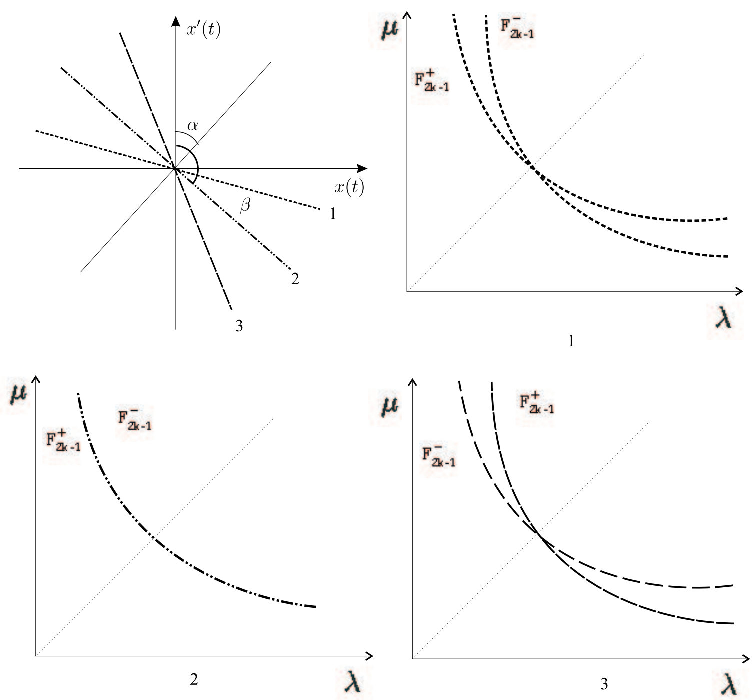

Figure 3: Comparison of asymptotes of the branches F^+_{2k-1} and F^-_{2k-1}.

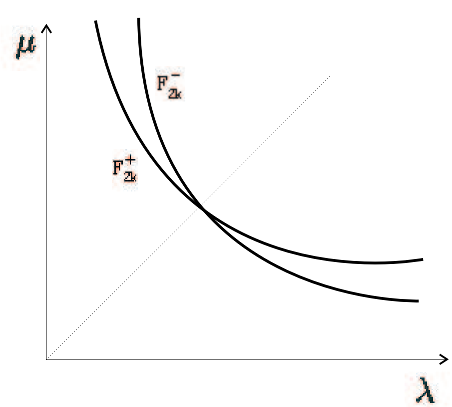

Figure 4: Comparison of asymptotes of the branches F^+_{2k} and F^-_{2k}.

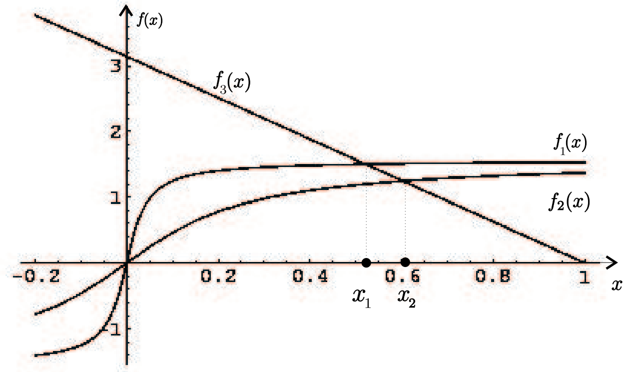

Figure 5: The functions $f_3 = \pi(1-x)$, $f_1 = \arctan(30x)$ and $f_2 = \arctan(5x)$ in the case of $\omega_1 = \arctan 30 > \omega_2 = \arctan 5$.

Proof. We discuss the case of $\alpha > \gamma$. We show first that $\lambda^-_{2k-1} < \lambda^+_{2k-1}$.

The asymptotes λ^-_{2k-1}, λ^+_{2k-1} are solutions of

$$\pi(k-z) = \arctan(z \tan\omega), \tag{12}$$

when $\omega_1 = \alpha$ and $\omega_2 = \gamma$ respectively. We consider the equation (12) in the case, when $\omega_1 > \omega_2$.

From the fact that $\alpha, \gamma \in (0; \frac{\pi}{2})$, $z > 0$, we obtain that

$$\tan\omega_1 > \tan\omega_2 \text{ and } \arctan(z\omega_1) > \arctan(z\omega_2).$$

Hence the corresponding roots z_1 and z_2 relate as $z_1 < z_2$ (see Fig. 2.1).

From this

$$\lambda^-_{2k-1} < \lambda^+_{2k-1}.$$

The inequality $\mu^-_{2k-1} > \mu^+_{2k-1}$ is true, since F^+_{2k-1} and F^-_{2k-1} are symmetrical.

In the case of $\alpha < \gamma$ the proof is similar and give that

$$\lambda^-_{2k-1} > \lambda^+_{2k-1}, \quad \mu^-_{2k-1} < \mu^+_{2k-1}.$$

We will show that the inequality $\lambda^-_{2k} > \lambda^+_{2k}$ for $\forall\alpha,\ \gamma$ holds. We recall, that λ^-_{2k} is the solution of

$$\pi[(1+k) - \lambda^-_{2k}] = \arctan(\lambda^-_{2k}\tan\alpha) + \arctan(\lambda^-_{2k}\tan\gamma),$$

or

$$\pi k - \pi\lambda^-_{2k} = \arctan(\lambda^-_{2k}\tan\alpha) + \arctan(\lambda^-_{2k}\tan\gamma) - \pi.$$

Since $\arctan(z\tan\varphi) < \frac{\pi}{2}$, if $\varphi \neq \frac{\pi}{2} + 2\pi n,\ n \in \mathbb{N}$, then

$$\arctan(\lambda^-_{2k}\tan\alpha) + \arctan(\lambda^-_{2k}\tan\gamma) - \pi < \frac{\pi}{2} + \frac{\pi}{2} - \pi = 0,$$

and

$$\pi k - \pi\lambda^-_{2k} < 0.$$

Thus $\lambda^+_{2k} = k < \lambda^-_{2k}$.

The inequality $\mu^-_{2k} < \mu^+_{2k}$ holds, since F^+_{2k} and F^-_{2k} are symmetrical.

3 Specific cases

3.1 Dirichlet problem

The Dirichlet problem is the specific case of the problem (1), (2), when $\alpha = 0$ un $\beta = \pi$

$$\begin{cases} x'' = -\mu^2 x^+ + \lambda^2 x^-, \\ x(0)\cos 0 - x'(0)\sin 0 = 0, \\ x(\pi)\cos\pi - x'(\pi)\sin\pi = 0. \end{cases}$$

Let us use the polar coordinates, then

$$\begin{cases} \varphi' = \begin{cases} \mu^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi \geq 0, \\ \lambda^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi < 0, \end{cases} \\ \varphi(0) = 0, \\ \varphi(\pi) = \pi + \pi n \end{cases}$$

for some integer n. From Theorem 2.1 we have the expressions for the Fuchik spectrum for the Dirichlet problem

$$\begin{aligned} F_{2k}^{+}: &\quad \frac{(k+1)\pi}{\mu} + \frac{k\pi}{\lambda} = \pi, \\ F_{2k}^{-}: &\quad \frac{k\pi}{\mu} + \frac{(k+1)\pi}{\lambda} = \pi, \\ F_{2k-1}^{\pm}: &\quad \frac{k\pi}{\mu} + \frac{k\pi}{\lambda} = \pi, \qquad k = 0,\ 1,\ 2,\ \ldots. \end{aligned}$$

Dividing by π we have the expressions for the Fuchik spectrum for the Dirichlet problem, who are the identical with the result of Theorem of [[1], p. 244].

The spectrum is shown in Fig. 6.

3.2 The boundary conditions $\alpha = \pi/4, \quad \beta = 3\pi/4$

We consider the problem for $\alpha = \dfrac{\pi}{4}$ and $\beta = \dfrac{3\pi}{4}$

$$\begin{cases} x'' = -\mu^2 x^+ + \lambda^2 x^-, \\ x(0)\cos\frac{\pi}{4} - x'(0)\sin\frac{\pi}{4} = 0, \\ x(\pi)\cos\frac{3\pi}{4} - x'(\pi)\sin\frac{3\pi}{4} = 0. \end{cases}$$

In polar coordinates

$$\begin{cases} \varphi' = \begin{cases} \mu^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi \geq 0, \\ \lambda^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi < 0, \end{cases} \\ \varphi(0) = \frac{\pi}{4}, \\ \varphi(\pi) = \frac{3\pi}{4} + \pi n \end{cases}$$

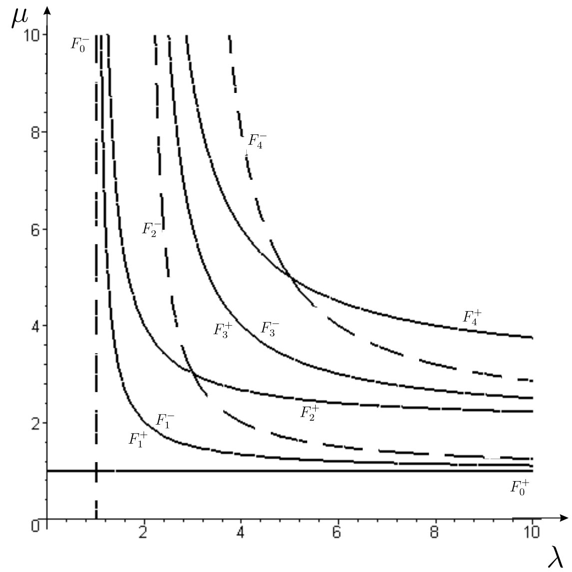

Figure 6: The Fuchik spectrum for Dirichlet problem, $\alpha = 0$ and $\beta = \pi$.

for some integer n. From Theorem 2.1 we have the expressions for the Fuchik spectrum for this problem

$$F_0^+ : \quad \frac{\pi}{\mu} - 2\frac{\arctan\mu}{\mu} = \pi,$$

$$F_0^- : \quad \frac{\pi}{\lambda} - 2\frac{\arctan\lambda}{\lambda} = \pi,$$

$$F_1^\pm : \quad \frac{\pi}{\mu} - \frac{\arctan\mu}{\mu} + \frac{\pi}{\lambda} - \frac{\arctan\lambda}{\lambda} = \pi,$$

$$F_2^+ : \quad \frac{2\pi}{\mu} - 2\frac{\arctan\mu}{\mu} + \frac{\pi}{\lambda} = \pi,$$

$$F_2^- : \quad \frac{2\pi}{\lambda} - 2\frac{\arctan\lambda}{\lambda} + \frac{\pi}{\mu} = \pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_{2k-1}^\pm : \quad \frac{k\pi}{\mu} - \frac{\arctan\mu}{\mu} + \frac{k\pi}{\lambda} - \frac{\arctan\lambda}{\lambda} = \pi,$$

$$F_{2k}^+ : \quad \frac{(k+1)\pi}{\mu} - 2\frac{\arctan\mu}{\mu} + \frac{k\pi}{\lambda} = \pi,$$

$$F_{2k}^- : \quad \frac{(k+1)\pi}{\lambda} - 2\frac{\arctan\lambda}{\lambda} + \frac{k\pi}{\mu} = \pi, \quad \forall k \in \mathbb{N}.$$

The spectrum is shown in Fig. 7.

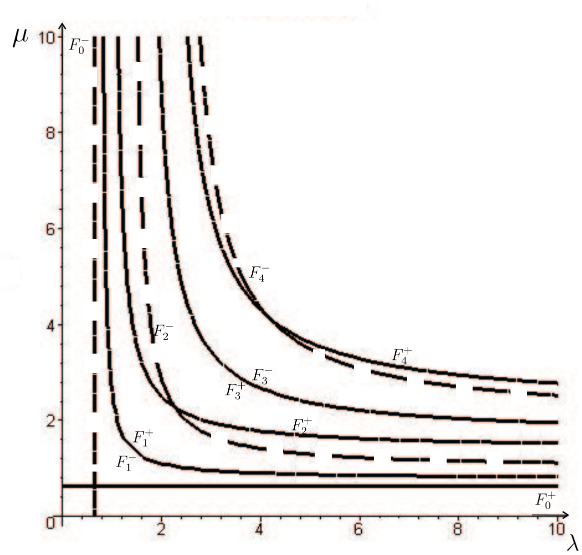

Figure 7: The Fuchik spectrum for the case of $\alpha = \frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$.

3.3 Neumann problem

Finally we consider the case of $\alpha = \dfrac{\pi}{2}$ and $\beta = \dfrac{\pi}{2}$

$$\begin{cases} x'' = -\mu^2 x^+ + \lambda^2 x^-, \\ x(0)\cos\frac{\pi}{2} - x'(0)\sin\frac{\pi}{2} = 0, \\ x(\pi)\cos\frac{\pi}{2} - x'(\pi)\sin\frac{\pi}{2} = 0, \end{cases}$$

or

$$\begin{cases} x'' = -\mu^2 x^+ + \lambda^2 x^-, \\ x'(0) = x'(\pi) = 0. \end{cases}$$

In polar coordinates

$$\begin{cases} \varphi' = \begin{cases} \mu^2 \sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi \geq 0, \\ \lambda^2 \sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi < 0, \end{cases} \\ \varphi(0) = \frac{\pi}{2}, \\ \varphi(\pi) = \frac{\pi}{2} + \pi n \end{cases}$$

for some integer n. From Theorem 2.1 we have the expressions for the Fuchik spectrum for the Neumann problem

$$\begin{array}{ll} F_0^+ : & \mu = 0, \\ F_0^- : & \lambda = 0, \\ F_1^\pm : & \dfrac{1}{2\mu} + \dfrac{1}{2\lambda} = 1, \\ F_2^\pm : & \dfrac{1}{\mu} + \dfrac{1}{\lambda} = 1, \end{array}$$

....

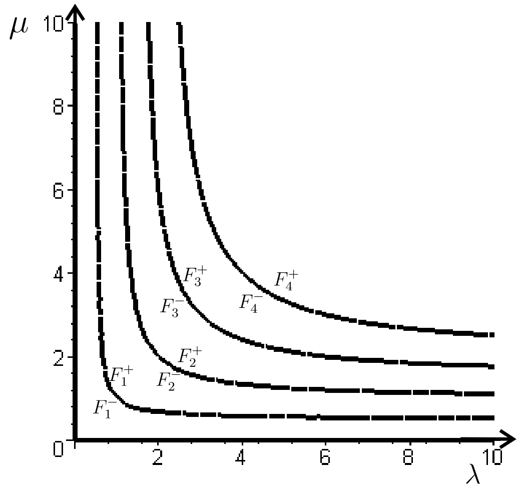

Figure 8: The Fuchik spectrum for the case of Neumann problem, $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$.

$$F_n^{\pm} : \qquad \frac{n}{2\mu} + \frac{n}{2\lambda} = 1, \quad \forall n \in \mathbb{N}.$$

The spectrum of the Neumann problem is shown in Fig. 8.

References

[1] A. Kufner and S. Fučik. Nonlinear differential equations(in Russian). - Russian Edition: Moscow, Nauka, 1988. - 304 p.

[2] T. Garbuza, On the Fuchik spectra, Daugavpils University, *Master work,* 2005, 68 p.

Т. Гарбуза. Спектр Фучика для краевой задачи второго порядка с условиями Штурма-Лиувилля.

Аннотация. Изучается задача на собственные значения. Рассматривается кусочно-линейное дифференциальное уравнение второго порядка с краевыми условиями Штурма - Лиувилля

$$x'' = -\mu^2 x^+ + \lambda^2 x^-,$$

$$\mu,\ \lambda \in \mathbb{R}, \quad x^{\pm}(t) = \max\{\pm x, 0\},$$

$$\begin{cases} x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0. \end{cases}$$

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T. Garbuza. Fučika spektrs otrās kārtas robežproblēmai ar Šturma - Liuvila robežnosacījumiem.

Anotācija. Tiek pētīts uzdevums par īpašvērtībām. Tiek apskatīts otrās kārtas gabaliem lineārs diferenciālvienādojums ar Šturma-Liuvila robežnosacījumiem

$$x'' = -\mu^2 x^+ + \lambda^2 x^-,$$

$$\mu,\ \lambda \in \mathbb{R}, \quad x^{\pm}(t) = \max\{\pm x, 0\},$$

$$\begin{cases} x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0. \end{cases}$$

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