Multiplicity results for the Neumann boundary value problem

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Summary. We provide multiplicity results for the Neumann boundary value problem where the second order differential equation is of the form $x'' = f(x)$.

1991 MSC 34B15

1 Introduction

We consider equation

$$x'' = f(x), \tag{1}$$

where $f(x)$ is a continuously differentiable function which has 5 simple zeros, together with the boundary conditions

$$x'(0) = 0, \quad x'(1) = 0. \tag{2}$$

Our goal is to get the multiplicity results for the problem (1), (2). In our considerations we use the phase plane analysis. Our results can be generalized to the case of $f(x)$ being a function with n simple zeros.

2 Simple cases

Our assumptions on a function $f(x)$ are:

(C1) $f \in C^1(R)$;

(C2) $f(x)$ has simple zeros at $p_1 < p_2 < p_3 < p_4 < p_5$;

(C3) $f(-\infty) = -\infty$ and respectively $f(+\infty) = +\infty$.

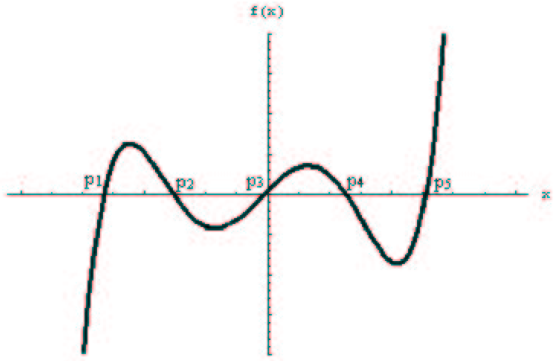

Figure 2.1 The function $f(x)$.

Let us consider the primitive $F(x) = \int_0^x f(s)\,ds$.

The function $F(x)$ has exactly 3 local minimums at the points $p_1 < p_3 < p_5$ and consequently 2 local maximums at the points $p_2 < p_4$ as is shown in Fig. 2.2.

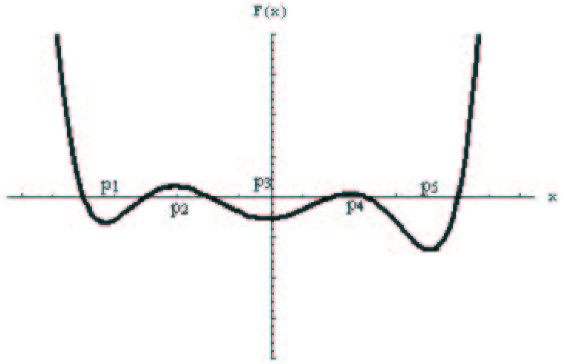

Figure 2.2 The primitive $F(x)$.

The phase portrait of the equivalent system

$$\begin{cases} x' &= y, \\ y' &= f(x) \end{cases} \tag{3}$$

depends on properties of the function $f(x)$ and its primitive $F(x)$.

There are exactly 2 critical points of the type "center" at $(p_2; 0)$, $(p_4; 0)$ and exactly 3 critical points of the type "saddle" at $(p_1; 0)$, $(p_3; 0)$, $(p_5; 0)$.

Let us consider the cases:

1. $F(p_1) < F(p_3) < F(p_5)$,
2. $F(p_5) < F(p_3) < F(p_1)$,
3. $F(p_3) < F(p_1) < F(p_5)$,
4. $F(p_3) < F(p_5) < F(p_1)$.

We consider the case 1 for definiteness. The following phase portrait describes solutions of the system (3).

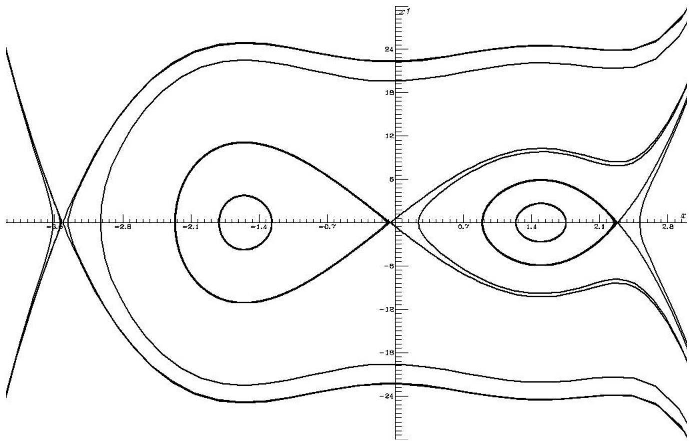

Figure 2.3 The phase plane of the case 1.

Theorem 2.1 *Let the conditions*

$$n^2\pi^2 < |f_x(p_2)| < (n+1)^2\pi^2, \tag{4}$$

$$m^2\pi^2 < |f_x(p_4)| < (m+1)^2\pi^2 \tag{5}$$

hold. Then the Neumann boundary value problem (1), (2) has at least $2n+2m$ *nontrivial solutions.*

Before to prove the theorem let us state the auxiliary results.

Lemma 2.1 *There exists the homoclinic solution emanating from the point* $(p_3; 0)$ *and going around the point* $(p_2; 0)$.

Proof. Consider the primitive $F_3(x) = \int_{p_3}^{x} f(s)\,ds$. Let r_1 be the first zero of $F_3(x)$ to the left of p_3. Consider the trajectory defined by the equation

$$x'^2 = 2F(x) - 2F(p_3) \tag{6}$$

and passing through the point $(r_1; 0)$. Let T_1 be the time needed for the point $(r_1; 0)$ to move to a position $(p_3; 0)$ along the trajectory. This time is given by the formula

$$T = \int_{r_1}^{p_3} \frac{ds}{\sqrt{2F(s) - 2F(p_3)}} \tag{7}$$

$$F(s) - F(p_3) = \frac{1}{2}F''(p_3)(s-p_3)^2 + \epsilon = \frac{1}{2}f'(p_3)(s-p_3)^2 + \epsilon.$$

$$\int_{r_1}^{p_3} \frac{ds}{\sqrt{f'(p_3)(s-p_3)^2}} = \frac{1}{\sqrt{f'(p_3)}} \int_{r_1}^{p_3} \frac{d(s-p_3)}{\sqrt{(s-p_3)^2}} =$$

$$= \frac{1}{\sqrt{f'(p_3)}} \int_{r_1-p_3}^{0} \frac{d\varepsilon}{\varepsilon} = \frac{1}{\sqrt{f'(p_3)}} \ln \varepsilon\big|_{r_1-p_3}^{0} = +\infty.$$

It is seen from (6) that any trajectory of equation (1) is symmetric with respect to the x-axis. □

Lemma 2.2 *There exists the homoclinic solution emanating from the point* $(p_5; 0)$ *and going around the point* $(p_4; 0)$.

Proof. Consider the primitive $F_5(x) = \int_{p_5}^{x} f(s)\, ds$. Let r_2 be the first zero of $F_5(x)$ to the left of p_5. Consider the trajectory defined by the equation

$$x'^2 = 2F(x) - 2F(p_5) \tag{8}$$

and passing through the point $(r_2; 0)$. Let T_2 be the time needed for the point $(r_2; 0)$ to move to a position $(p_5; 0)$ along the trajectory. This time is given by the formula

$$T = \int_{r_2}^{p_5} = \frac{ds}{\sqrt{2F(s) - 2F(p_5)}} \tag{9}$$

$$F(s) - F(p_5) = \frac{1}{2}F''(p_5)(s - p_5)^2 + \epsilon = \frac{1}{2}f'(p_5)(s - p_5)^2 + \epsilon.$$

$$\int_{r_2}^{p_5} \frac{ds}{\sqrt{f'(p_5)(s - p_5)^2}} = \frac{1}{\sqrt{f'(p_5)}} \int_{r_2}^{p_5} \frac{d(s - p_5)}{\sqrt{(s - p_5)^2}} =$$
$$= \frac{1}{\sqrt{f'(p_5)}} \int_{r_2 - p_5}^{0} \frac{d\varepsilon}{\varepsilon} = \frac{1}{\sqrt{f'(p_5)}} \ln \varepsilon|_{r_2 - p_5}^{0} = +\infty.$$

Since any trajectory of equation (1) is symmetric with respect to the x-axis, the assertion follows. □

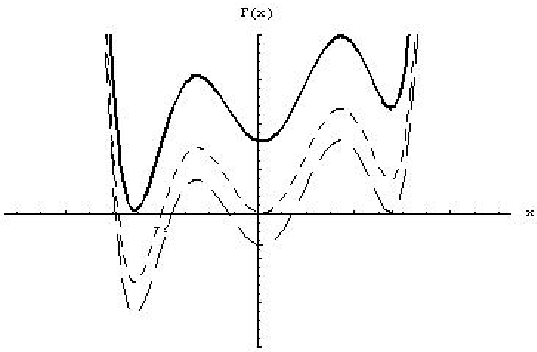

Figure 2.4 Three primitives given by
$F_1(x) = \int_{p_1}^{x} f(s)ds, \quad F_3(x) = \int_{p_3}^{x} f(s)ds, \quad F_5(x) = \int_{p_5}^{x} f(s)ds$

Proof. Let us prove the existence of multiple solutions going around the critical point $(p_2; 0)$. Let us consider the Cauchy problem (1),

$$x(0) = m, \quad x'(0) = 0, \tag{10}$$

where $m \in (r_1; p_2)$. If $m \sim p_2$, then solutions $x(t)$ of the problem (1), (10) behave like a solution of the linear problem

$$y'' = f_x(p_2)y, \quad y(0) = -1, \quad y'(0) = 0$$

and $T(m) \sim \frac{\pi}{\sqrt{|f_x(p_2)|}}$. If $n\,T(m) < 1 < (n + 1)\,T(m)$, then $T(m) \to +\infty$ for $m \to r_1$. Hence at least n solutions of the Neumann problem.

There are symmetrical solutions for the case $m \in (p_2; p_3.)$ Totally at least $2n$ solutions.

Then we consider the second critical point $(p_4; 0)$. Similar considerations yield at least $2m$ solutions going around the fourth critical point $(p_4; 0)$, the second "center" type point.

Remark. Theorem 2.1 is valid also for the cases 2, 3, 4. The respective proofs can be carried out.

3 More complicated cases

Let us consider the cases

1. $F(p_1) < F(p_5) < F(p_3)$,
2. $F(p_5) < F(p_1) < F(p_3)$.

These cases are symmetrical and we consider only the first one.

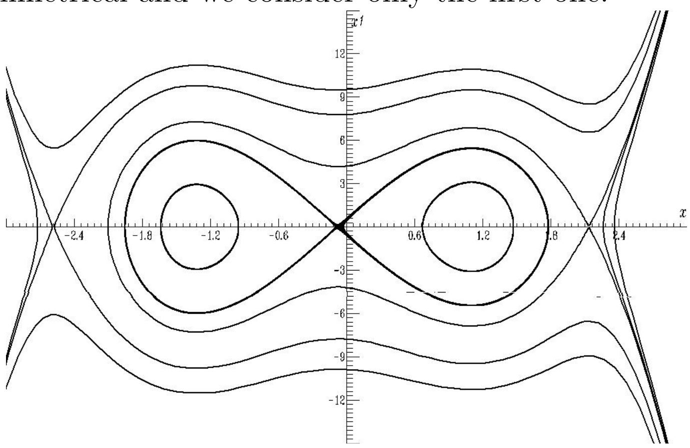

Figure 3.1 The phase portrait for the case of $F(p_1) < F(p_5) < F(p_3)$.

Consider the function

$$T(x_0) = \frac{1}{\sqrt{2}} \int_{x_0}^{x_1(x_0)} \frac{ds}{\sqrt{F(s) - F(x_0)}},$$

which is defined in the interval $(x_0^*; x_0^{**})$, where $x_1(x_0)$ is the first zero to the right of x_0 of the function $F(s) - F(x_0)$.

Define x_0^* to be the first zero to the left of p_5 of the function $F_5 = \int_{p_5}^{x} f(s)ds$.

Define x_0^{**} to be the first zero to the left of p_3 of the function $F_3 = \int_{p_3}^{x} f(s)ds$. Obviously $p_1 < x_0^* < x_0^{**} < p_2$.

Lemma 3.1 *Let $T_{min} = \min\{T(x_0) : \ x \in (x_0^*, x_0^{**})\}$, where x_1 is the first zero of the function $F(s) - F(x_0)$ to the right of x_0. Suppose that there exists an integer k such that $k\,T_{min} < 1 < (k+1)\,T_{min}$.*

Then there are at least $4k$ solutions of the Neumann boundary value problem, with trajectories going around the two singular points of the type "center".

Proof. Let $z : T(z) = min\{T(x_0), \, x_0^* < x_0 < x_0^{**}\}$.

Consider the Cauchy problem (1), $x(0) = x_0$, $x'(0) = 0$, $x_0 \in (x_0^*; z)$. When $x_0 \sim z$, then the half period $T(x_0)$ satisfies the condition $k\,T_{min} < 1 < (k+1)\,T_{min}$. On the other hand, $T(x_0) \to +\infty$ as $x_0 \sim x_0^*$. Hence at least k solutions of the problem. Similarly for the case $z < m < x_0^{**}$. Hence additionally at least k solutions.

Define z_1 as the first zero of $F(s) - F(z)$ to the right of z. Notice that $z_1 \in (z_2; p_5)$. Consider the problem $x(0) = n$ for $n \in (z_2; z_1)$. When $n \sim z_1$, the condition is satisfied, and $T(n) \to +\infty$ as $n \sim z_2$. Hence at least k solutions of the problem. Similarly for $n \in (z_1; p_5)$.

Totally at least $4k$ solutions.

Theorem 3.1 *Let the conditions (4) and (5) hold. Suppose that there exists an integer k such that*

$$kT_{min} < 1 < (k+1)T_{min}.$$

Then the Neumann boundary value problem (1), (2) has at least $2n + 2m + 4k$ nontrivial solutions.

The proof follows from the proofs of Theorem 2.1 and Lemma 3.1.

4 Examples

Example 1. Consider the second-order nonlinear boundary value problem

$$\begin{aligned} &x'' = 6x^5 + 55x^4 + 35x^3 - 408.75x^2 + 33x + 369, \\ &x'(0) = x'(1) = 0. \end{aligned} \tag{11}$$

The function $f(x) = 6x^5 + 55x^4 + 35x^3 - 408.75x^2 + 33x + 369$ has exactly 3 critical points of the type "saddle" at the points $p_1 = -6.90$, $p_3 = -0.92$, $p_5 = 1.80$ and 2 critical points of the type "center" at the point $p_2 = -4.37$, $p_4 = 1.22$.

Respectively the function

$$F(x) = x^6 + 11x^5 + 8.75x^4 - 136.25x^3 + 16.5x^2 + 369x - 270$$

has 3 local minimums and consequently 2 local maximums as is shown in Fig. 4.1.

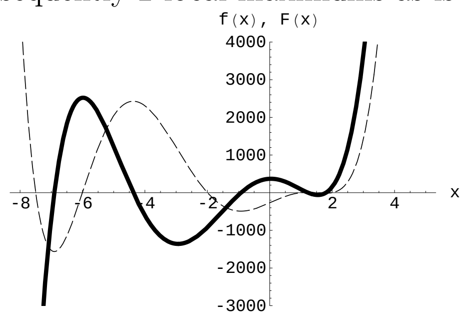

Figure 4.1 The function $f(x)$ and its primitive $F(x)$

$$f_x(p_2) = -1811.24;$$

The condition

$$13^2\pi^2 < 1811.24 < 14^2\pi^2$$

holds. Then the boundary value problem (11) has at least 26 solutions.

$$f_x(p_4) = -337.374;$$

The condition

$$5^2\pi^2 < 337.374 < 6^2\pi^2,$$

holds. Then the boundary value problem (11) has at least 10 solutions.

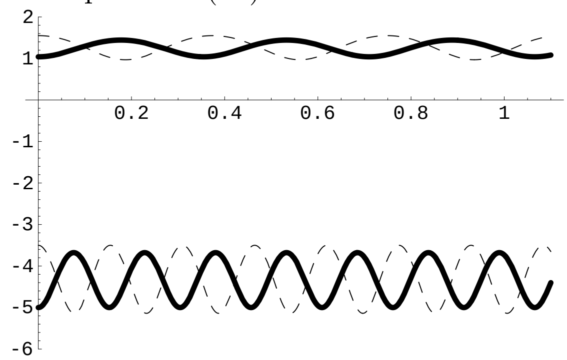

Figure 4.2

Hence the boundary value problem (11) has at least 36 solutions.

Example 2. Consider the second-order nonlinear boundary value problem

$$\begin{aligned} &x'' = 6x^5 - 2.5x^4 - 138x^3 - 34.5x^2 + 247x + 12 \\ &x'(0) = x'(1) = 0. \end{aligned} \tag{12}$$

The function $f(x) = 6x^5 - 2.5x^4 - 138x^3 - 34.5x^2 + 247x + 12$ has exactly 3 critical points of the type "saddle" at the points $p_1 = -4.19$, $p_3 = -0.05$, $p_5 = 4.95$ and 2 critical points of the type "center" at the point $p_2 = -1.57$, $p_4 = 1.27$.

Respectively the function

$$F(x) = x^6 - 0.5x^5 - 34.5x^4 - 11.5x^3 + 123.5x^2 + 12x - 90$$

has 3 local minimums and consequently 2 local maximums as is shown in Fig. 4.3.

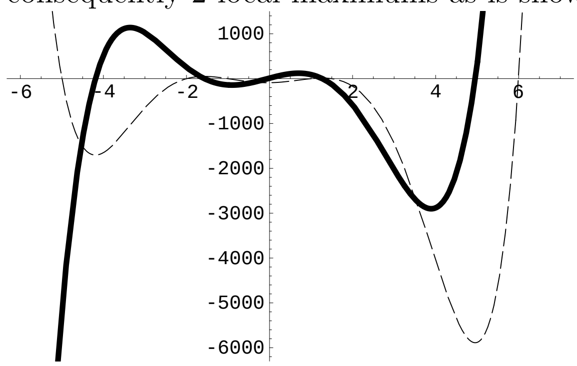

Figure 4.3 The function $f(x)$ and its primitive $F(x)$

$$f_x(p_2) = -442.928;$$

The condition

$$6^2\pi^2 < 442.928 < 7^2\pi^2,$$

holds. Then the boundary value problem (12) has at least 12 solutions.

$$f_x(p_4) = -452.215;$$

The condition

$$6^2\pi^2 < 452.215 < 7^2\pi^2,$$

holds. Then the boundary value problem (12) has at least 12 solutions.

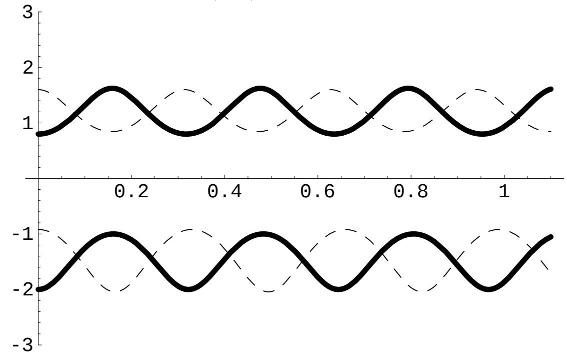

Figure 4.4

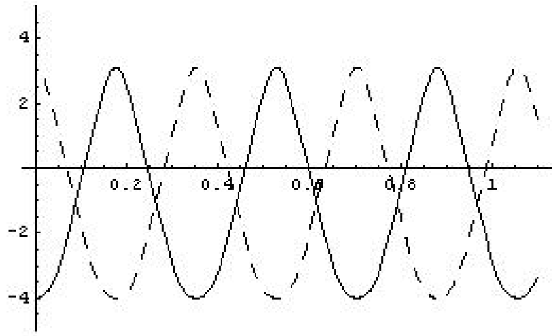

Figure 4.5

Hence the boundary value problem (11) has at least 44 solutions.

References

[1] S. Ogorodnikova and F. Sadyrbaev. Estimations ot the number of solutions to some nonlinear second order boundary value problems - Mathematics. Differential Equations (Univ. of Latvia,Institute of Math. and Comp. Sci.), vol. 5 (2005), pp. 24–32.

С. Атслега. Результаты о числе решений для задачи Неймана.

Аннотация. Приводятся результаты о числе решений в задаче Неймана для дифференциального уравнения второго порядка вида $x'' = f(x)$.

УДК 517.927

S. Atslega. Rezultāti par atrisinājumu skaitu Neimana problēmā.

Anotācija. Sniegti rezultāti par atrisinājumu skaitu Neimana problēmā otrās kārtas diferenciālvienādojumam formā $x'' = f(x)$.

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Received 02.05.06