Types of solutions and multiplicity results for two-point fourth order nonlinear boundary value problems

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Summary. Two-point boundary value problems for the fourth order ordinary nonlinear differential equations with monotone right sides are considered. If the respective nonlinear equation can be reduced to a quasi-linear one with a non-resonant linear part and both equations are equivalent in some domain D, and if solutions of the quasi-linear problem lie in D, then the original problem has a solution. We say then that the original problem allows for quasilinearization. We show that a quasi-linear problem has a solution of definite type which corresponds to the type of the linear part. If quasilinearization is possible for essentially different linear parts, then the original problem has multiple solutions.

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1 Introduction

Consider the boundary value problem

$$x^{(4)} = f(t, x, x'), \quad t \in I := [0, 1], \tag{1}$$

$$x(0) = x'(0) = 0 = x(1) = x'(1). \tag{2}$$

Function $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is supposed to be continuous together with the partial derivatives f_x and $f_{x'}$ (the unique solvability of the Cauchy problem $x(0) = x_0,\ x'(0) = y_0$ is ensured as well as the continuous dependence of solutions on initial data). Consider also the quasi-linear equation

$$x^{(4)} = k^4 x + F(t, x, x'), \tag{3}$$

where $F, F_x, F_{x'} : I \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and F is bounded, that is, $|F(t, x, x')| < M$ $\forall (t, x, x') \in I \times \mathbb{R}^2$. If the linear part $(L_4 x)(t) := x^{(4)} - k^4 x$ is nonresonant with respect to the boundary conditions (2), that is, the homogeneous problem $(L_4 x)(t) = 0$, (2) has only the trivial solution, then the problem (3), (2) is solvable. Suppose that equations

(1) and (3) are equivalent in a domain $D = \{(t, x, x');\ 0 \le t \le 1,\ |x| \le N,\ |x'| \le N_1\}$. If any solution $x(t)$ of the problem (3), (2) satisfies the estimates

$$|x(t)| \le N, \quad |x'(t)| \le N_1, \quad \forall t \in I, \tag{4}$$

then it solves also the problem (1), (2). We will say for brevity that the problem (1), (2) allows for quasilinearization with respect to the linear part $(L_4 x)(t)$.

Suppose that the problem (1), (2) allows for quasilinearization with respect to a different linear part $(l_4 x)(t)$ also. Does that mean that the original problem has another solution, generated by this quasilinearization?

In what follows we try to answer this question.

Our research is motivated by the papers of L. Erbe [3], H. Knobloch [6], [7], L. Jackson and K. Schrader [5], who studied oscillatory properties of solutions of two-point second order boundary value problems. They characterized a solution of BVP by oscillatory properties of the respective linear equation of variations

$$y'' = F_x(t, \xi, \xi')y + F_{x'}(t, \xi, \xi')y'. \tag{5}$$

It was proved essentially in the mentioned results that the BVP for a quasilinear equation $x'' = F(t, x, x')$ with a bounded F has a solution $\xi(t)$, for which equation (5) is disconjugate in the interval $[0, 1]$, that is, a solution $y(t)$, which is defined by the initial conditions $y(0) = 0, \quad y'(0) = 1$, does not vanish for $t \in (0, 1)$ (it may vanish at $t = 1$, however).

It was shown by the authors in [13] that similar results are valid for quasi-linear problems for equations of the form

$$x'' + p(t)x' + q(t)x = F(t, x, x'). \tag{6}$$

The Dirichlet problem for equation above must have a solution $\xi(t)$, which induces the same oscillatory properties for the respective equation of variations, as the linear part in (6) has.

This general result was applied then to show that for some equations the process of quasilinearization may prove the existence of multiple solutions.

The aim of this article is to generalize the described results on quasilinearization to the fourth order quasi-linear and then nonlinear equations.

We provide first basics of the oscillatory theory for two-termed linear fourth order differential equations. This theory was developed by Leighton and Nehari [8].

Definitions of the type of a solution to a fourth order boundary value problems are given.

The general result and related auxiliary results for quasi-linear problems are stated.

The idea of quasilinearization is described, which is applied to investigation of the Emden – Fowler fourth order nonlinear equation.

Finally the results of computation are provided, which show that multiple solutions of different types can be obtained, using the quasilinearization process, for the Dirichlet boundary value problem for the Emden – Fowler type equation.

It should be mentioned that, as a by-product, estimates of the Green's function for the fourth order linear boundary value problem were obtained.

2 Preliminaries

In this section we provide basics of the oscillation theory by Leighton and Nehari for equations of the form

$$x^{(4)} = p(t)x, \tag{7}$$

where $p(t)$ is a positive valued continuous function.

Definition 1. A point η is called by *a conjugate point* for the point $t = 0$, if there exists a nontrivial solution $x(t)$ such that

$$x(0) = x'(0) = 0 = x(\eta) = x'(\eta).$$

The main result in [8] is stated below.

Theorem 2.1 *Suppose that equation (7) has a solution with $n+3$ zeros for $t > 0$. Then there exist n conjugate to $t = 0$ points η_i, which form ascending sequence*

$$\eta_0 < \eta_1 < \ldots < \eta_{n-1}.$$

The respective solutions $x_i(t)$ $(i = 0, 1, \ldots, n-1)$, which are called by extremal solutions, have exactly i simple zeros in the intervals $(0, \eta_i)$.

Corollary 2.1 *The equation $x^{(4)} = k^4 x$ has infinite sequence of conjugate (to the point $t = 0$) points η_i.*

Proof. Since the equation above has a nontrivial solution with infinitely many zeros, the assertion follows. □

Suppose that initial conditions are of the form

$$x(0) = x'(0) = 0, \tag{8}$$

$$x''(0) = r\cos\Theta, \quad x'''(0) = r\sin\Theta. \tag{9}$$

It was shown in [8] that no extremal solutions are possible for $\Theta \in [0, \pi/2]$ and $\Theta \in [\pi, 3\pi/2]$. Let Θ_i relate to an extremal solution $x_i(t)$. It was shown in [11] that Θ_i are arranged as follows for solutions with positive $x''(0)$ (and respectively negative $x'''(0)$)

$$-\pi/2 < \Theta_2 < \ldots < \Theta_{2n} < \ldots < \Theta_{2m+1} < \ldots < \Theta_1 < 0. \tag{10}$$

Theorem 2.2 ([11]) *Conjugate points continuously depend on the coefficient $p(t)$.*

3 Quasi-linear problems

Consider the quasi-linear equation

$$x^{(4)} - k^4 x = F(t, x) \tag{11}$$

together with the boundary conditions (2). Suppose the following conditions are satisfied.

(A1) F and F_x are $C(I \times R)$-functions;

(A2) $F(t,0) \equiv 0$;

(A3) $k^4 + F_x(t,x) > 0$ for any $(t,x) \in I \times \mathbb{R}$.

Definition 2. We will say that the linear part $(L_4x)(t) := x^{(4)} - k^4 x$ is i-nonresonant with respect to the boundary conditions (2), if there are exactly i conjugate points in the interval $(0,1)$ and $t = 1$ is not a conjugate point.

Definition 3. We will say that $\xi(t)$ is an i-type solution of the problem (11), (2), if for small enough $\alpha, \beta > 0$ the difference $u(t;\alpha,\beta) = x(t;\alpha,\beta) - \xi(t)$ has exactly i conjugate points in $(0,1)$ and $t = 1$ is not a conjugate point, where $x(t;\alpha,\beta)$ is a solution of (11), which satisfies the initial conditions

$$x(0;\alpha,\beta) = \xi(0), \quad x'(0;\alpha,\beta) = \xi'(0), \tag{12}$$

$$x''(0;\alpha,\beta) - \xi''(0) = \alpha, \quad x'''(0;\alpha,\beta) - \xi'''(0) = -\beta. \tag{13}$$

Remark 3.1. An i-type solution ξ of the problem (11), (2) has the following characteristics in terms of the variational equation: a solution $y(t)$ of the respective variational equation either has exactly i conjugate points in the interval $(0,1]$, or it has exactly i conjugate points in the interval $(0,1)$ and $t = 1$ is a conjugate point. The cases of the i-th conjugate point of being at $t = 1$ or $(i+1)$-th conjugate point being at $t = 1$ are not excluded.

Theorem 3.1 *Quasi-linear problem (11), (2) with an i-nonresonant linear part $(L_4x)(t)$ has an i-type solution.*

We state several lemmas before to prove the theorem.

Lemma 3.1 *A set S of all solutions of the BVP (11), (2) is non-empty and compact in* $C^3([0,1])$.

Proof. Solvability can be proved by standard application of the Schauder principle to the operator $T : C^3(I) \to C^3(I)$, where T is defined by

$$(Tx)(t) = \int_0^1 G(t,s) F(s, x(s))\, ds$$

and $G(t,s)$ is the Green's function for $(L_4x)(t) = 0$, (2). Notice that F is bounded.

Compactness of S is obtained by routine application of the Arzela - Ascoli criterium.

Remark 3.2. Solvability of quasi-linear problems with nonresonant linear parts is well-known ([2]).

Remark 3.3. Any solution $x(t)$ of the problem (11), (2) satisfies the estimate

$$\max_I |x(t)| \leq \Gamma \cdot M, \tag{14}$$

where $\Gamma = \max\limits_{0 \leq t,s \leq 1} |G(t,s)|$, $M = \sup\{|F(t,x)| : (t,x) \in I \times \mathbb{R}\}$.

Lemma 3.2 *There are elements $x^*(t)$ and $x_*(t)$ in S, which possess the properties:* $x^{*\prime\prime 2}(0) + x^{*\prime\prime\prime 2}(0) = \max\{x''^2(0) + x'''^2(0) :\ x \in S,\ x''(0) > 0,\ x'''(0) < 0\}$. *Similarly for* x_*.

Proof. The set $S_1 = \{x''^2(0) + x'''^2(0) : x \in S\}$ is an image of a continuous map $M : C^3([0,1]) \to \mathbb{R}$ defined by $M(x) = x''^2(0) + x'''^2(0)$.

Denote by $x(t; r, \Theta)$ a solution of the Cauchy problem (11), (8), (9).

Lemma 3.3 *If $x(t; r, \Theta)$ is a nontrivial solution of the BVP (11), (2), then either $\Theta \in (-\pi/2, 0)$ or $\Theta \in (\pi/2, \pi)$.*

Proof. If this were not the case then either $x''(0) \geq 0$ and $x'''(0) \geq 0$ or $x''(0) \leq 0$ and $x'''(0) \leq 0$. Taking into account that $x(0) = x'(0) = 0$ one concludes that either $x(t) \geq 0$ together with the derivatives of order up to the third, or, respectively, $x(t) \leq 0$ along with the derivatives. If, say, $x^{(i)}(t) \geq 0$, $(i = 0, 1, 2, 3)$ then either $x(t) \equiv 0$ or it does not satisfy the boundary conditions at $t = 1$.

Lemma 3.4 *Let the conditions **(A1)** – **(A3)** be fulfilled. Suppose that the linear part $(L_4x)(t)$ in (11) is i-nonresonant. Let ξ be any element of S.*

Then the function $u(t; r, \Theta) = \dfrac{x(t; r, \Theta) - \xi(t)}{r}$ for any $\Theta \in [0, 2\pi)$ tends to a solution $y(t)$ of the Cauchy problem

$$y^{(4)} - k^4 y = 0, \quad y(0) = y'(0) = 0,\ y''(0) = \cos\Theta,\ y'''(0) = \sin\Theta \tag{15}$$

as $r \to +\infty$, where $x(t; r, \Theta)$ is a solution of the problem

$$x^{(4)} - k^4 x = F(t, x), \quad x(0) = x'(0) = 0,$$
$$x''(0) - \xi''(0) = r\cos\Theta,\ x'''(0) - \xi'''(0) = r\sin\Theta.$$

Proof. The functions $u(t; r, \Theta)$ solve the initial value problems

$$(L_4u)(t) = \frac{1}{r}[F(t, x(t)) - F(t, \xi(t)], \tag{16}$$

$$u(0) = u'(0) = 0, \quad u''(0) = \cos\Theta,\ u'''(0) = \sin\Theta.$$

Let $r \to +\infty$. The right side in (16) then tends to zero uniformly in t for fixed Θ. By classical results, $u(t; r, \Theta)$ tends then to a solution $y(t)$ of the problem (15).

Lemma 3.5 *Let ξ be any element of S and $x(t; r, \Theta)$ have the same meaning as in the precedent Lemma. The function $v = x - \xi$ satisfies then the equation*

$$(L_4v)(t) = \Phi(t)v, \tag{17}$$

where $\Phi(t; r, \Theta) = \dfrac{F(t, x) - F(t, \xi)}{x - \xi} = F_x(\omega(t))$ (ω, by Mean Value Theorem, stands for some intermediate point between ξ and x).

If equation (17) has for some r and Θ a conjugate point at $t = 1$ then the sum of the respective $v(t; r, \Theta)$ and ξ solves the BVP (11), (2).

Proof. Can be found in [11].

Proof of the Theorem 3.1. Let $\xi(t)$ be a solution $x^*(t)$ with the maximum property, described in Lemma 3.2. Consider solutions $x(t; r, \Theta)$ of the initial value problem (11),

$$x(0) = x'(0) = 0, \quad x''(0) - \xi''(0) = r\cos\Theta,\ x'''(0) - \xi'''(0) = r\sin\Theta. \tag{18}$$

The difference $u(t; r, \Theta) := x(t; r, \Theta) - \xi(t)$ satisfies the linear equation

$$(L_4)(t) = \Phi(t; r, \Theta)u, \tag{19}$$

where $\Phi(t; r, \Theta)$ is defined above. Consider linear equations (19) for $r \sim 0$. Suppose that $\xi(t)$ is not an i-type solution. To be definite, consider the case of the difference $u(t; r, \Theta)$ to have more than i points of double zero in the interval $(0, 1)$ for small values of r. Recall that $u'''(0) = r\sin\Theta < 0$ and $u''(0) = r\cos\Theta > 0$. Let $\Theta \in (-\frac{\pi}{2}, 0)$ be fixed. For $r \sim +\infty$ the respective linear equations (19) have exactly i conjugate points. Thus there exists $r_1(\Theta)$ such that the respective linear equation (19) has $(i+1)$-th conjugate point exactly at $t = 1$. This is true for any $\Theta \in (-\frac{\pi}{2}, 0)$. Consider the continuous curve $r_1(\Theta)$, which is defined for any $\Theta \in (-\frac{\pi}{2}, 0)$. Let $\omega(\Theta)$ be the respective angle.

Consider the difference $\omega(\Theta) - \Theta$. It has different signs for $\Theta = 0$ and $\Theta = -\dfrac{\pi}{2}$. Therefore there exists Θ_0 such that $\omega(\Theta_0) = \Theta_0$. Thus by Lemma 3.5 a solution to the BVP exists which has r greater than that for ξ. This contradicts the choice of $\xi = x^*$. Similarly other cases can be considered.

Thus ξ is an i-type solution of the problem (11), (2). Other cases can be treated similarly. □

4 Quasilinearization and multiple solutions

Consider an equation

$$x^{(4)} = f(t, x) \tag{20}$$

together with the boundary conditions (2).

Definition 4. Let equations (20) and (11), where the linear part $(L_4x)(t)$ in (11) is i-nonresonant in the interval I, be equivalent in a domain

$$D_N = \{(t, x): \quad 0 \le t \le 1, \quad |x| < N\} \tag{21}$$

in the sense that any solution $x : I \to \mathbb{R}$ of (20) with a graph in D_N is also a solution of (11) and vice versa. Suppose that any solution $x(t)$ of the quasi-linear problem (11), (2) satisfies an estimate

$$|x(t)| < N, \quad \forall t \in I. \tag{22}$$

We will say then that the problem (20), (2) allows for quasilinearization with respect to a domain D_N and a linear part $(L_4x)(t)$.

Remark 4.1. Any solution $x(t)$ of the problem (11), (2) satisfies the estimate (22) if the relation $\Gamma \cdot M < N$ holds, where Γ, M have the same meaning as in (14).

Theorem 4.1 *If the problem (20), (2) allows for quasilinearization with respect to some domain D_N and some i-nonresonant linear part $(L_4x)(t)$, then it has a solution.*

Proof. Let $x(t)$ be a solution of the quasi-linear problem (11), (2). If $x(t)$ satisfies the estimates (22) and equations (20) and (11) are equivalent in D_N, then $x(t)$ solves also the problem (20), (2).

Theorem 4.2 *Suppose that the problem (20), (2) allows for quasilinearization with respect to D_N and i-nonresonant linear part $(L_4x)(t)$, and, at the same time, it allows for quasilinearization with respect to a domain*

$$D_M = \{(t, x) : \quad 0 \leq t \leq 1, \quad |x| < M, \}$$

and j-nonresonant linear part $(l_4x)(t)$, where $i \neq j$.

Then the problem (20), (2) has at least 2 solutions.

Proof is evident.

Corollary 4.1 *Suppose that the problem (20), (2) allows for quasilinearization with respect to n essentially different (in the sense of Definition 4) linear parts and n domains of the form (21). Then it has at least n different solutions.*

5 Applications

Consider the boundary value problem for the forth-order differential equation

$$x^{(4)} = \alpha^2 \cdot |x|^p \operatorname{sign} x, \tag{23}$$

$$x(0) = x'(0) = 0 = x(1) = x'(1), \tag{24}$$

where $\alpha \neq 0, \quad p > 0, \quad p \neq 1$.
The equation (23) is equivalent to the equation

$$x^{(4)} - k^4 x = \alpha^2 \cdot |x|^p \operatorname{sign} x - k^4 x. \tag{25}$$

Suppose that k satisfies

$$\cos k \cdot \cosh k \neq 1. \tag{26}$$

The respective homogeneous problem

$$x^{(4)} - k^4 x = 0, \tag{27}$$

(24) then has only the trivial solution, that is, the linear part in (25) is non-resonant.

5.1 Green's function

The Green's function of the problem (27), (24) is given by

$$
\begin{aligned}
G_k(t,s) = \; & \tfrac{1}{\triangle}\Big(\big(e^{k(s+t)} + e^{2k-k(s+t)}\big)\sin k + \big(e^{k(t-s)} - e^{2k-k(t-s)}\big)\cos k + \\
& 2e^k \sin k(t-s) - e^{2k}\big(\cos ks - \sin ks\big)\big(\cos k(t-1) + \sin k(t-1)\big) + \\
& e^{k(t+1)}\big(\cos ks - \sin ks\big) + e^{k(s+1)}\big(\cos kt - \sin kt\big) - \\
& e^{kt}\big(\cos k(s-1) - \sin k(s-1)\big) - e^{ks}\big(\cos k(t-1) - \sin k(t-1)\big) - \\
& e^{k(1-t)}\big(\cos ks + \sin ks\big) - e^{k(1-s)}\big(\cos kt + \sin kt\big) + \\
& e^{k(2-t)}\big(\cos k(s-1) + \sin k(s-1)\big) + e^{k(2-s)}\big(\cos k(t-1) + \sin k(t-1)\big) + \\
& e^{k-k(t-s)} - e^{k+k(t-s)} + \big(\cos ks + \sin ks\big)\big(\cos k(t-1) - \sin k(t-1)\big)\Big), \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 0 \le s \le t \le 1,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
G_k(t,s) = \; & \tfrac{1}{\triangle}\Big(\big(e^{k(s+t)} + e^{2k-k(s+t)}\big)\sin k + \big(e^{k(s-t)} - e^{2k-k(s-t)}\big)\cos k + \\
& 2e^k \sin k(s-t) - e^{2k}\big(\cos kt - \sin kt\big)\big(\cos k(s-1) + \sin k(s-1)\big) + \\
& e^{k(t+1)}\big(\cos ks - \sin ks\big) + e^{k(s+1)}\big(\cos kt - \sin kt\big) - \\
& e^{kt}\big(\cos k(s-1) - \sin k(s-1)\big) - e^{ks}\big(\cos k(t-1) - \sin k(t-1)\big) - \\
& e^{k(1-t)}\big(\cos ks + \sin ks\big) - e^{k(1-s)}\big(\cos kt + \sin kt\big) + \\
& e^{k(2-t)}\big(\cos k(s-1) + \sin k(s-1)\big) + e^{k(2-s)}\big(\cos k(t-1) + \sin k(t-1)\big) + \\
& e^{k-k(s-t)} - e^{k+k(s-t)} + \big(\cos kt + \sin kt\big)\big(\cos k(s-1) - \sin k(s-1)\big)\Big), \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 0 \le t < s \le 1,
\end{aligned} \tag{29}
$$

where

$$\triangle = 8k^3 e^k(\cosh k \cos k - 1).$$

Proposition 5.1 *The Green's function $G_k(t,s)$ satisfies the estimate*

$$\big|G_k(t,s)\big| \le \Gamma_k = \frac{\cosh k + (1+\sqrt{2})(e^k+1)}{2k^3\big|\cosh k \cdot \cos k - 1\big|}. \tag{30}$$

5.2 Estimates

We wish to make the right side in (25) bounded. Denote

$$f_k(x) := \alpha^2 \cdot |x|^p \operatorname{sign} x - k^4 x.$$

The function $f_k(x)$ is odd. Let us consider it for nonnegative values of x. There exists a positive point of local extremum x_0 (it is either a point of minimum in case of $p > 1$ or a point of maximum in case of $0 < p < 1$),

$$x_0 = \Big(\frac{k^4}{\alpha^2 p}\Big)^{\frac{1}{p-1}}.$$

We can calculate the value of the function at the point of extremum x_0. Set

$$M_k = \big|f_k(x_0)\big| = \alpha^{\frac{2}{1-p}} \cdot \Big(\frac{k^4}{p}\Big)^{\frac{p}{p-1}} \cdot |p-1|. \tag{31}$$

Choose N_k so that

$$|x| \le N_k \Rightarrow |f_k(x)| \le M_k, \quad \forall\, t \in I.$$

The value of N_k is computed by solving the equation

$$f_k(x) = -f_k(x_0),$$

or, equivalently,

$$\alpha^2 \cdot |x|^p - k^4 x = \left(\frac{k^4}{p}\right)^{\frac{p}{p-1}} \cdot (1-p) \cdot \alpha^{\frac{2}{1-p}} \tag{32}$$

with respect to x. Computation gives

$$N_k = \left(\frac{k^4}{\alpha^2}\right)^{\frac{1}{p-1}} \beta, \tag{33}$$

where a constant β is to be found from the equation

$$\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}. \tag{34}$$

Equation (34) has a root $\beta > 1$ for any positive p ($p \neq 1$).

Let us consider the quasi-linear equation

$$x^{(4)} - k^4 x = F_k(x), \tag{35}$$

where

$$F_k(x) = \begin{cases} f_k(x), & |x| \leq N_k, \\ f_k(N_k + \varepsilon_1), & x \geq N_k + \varepsilon_1, \\ f_k(-N_k - \varepsilon_1), & x \leq -N_k - \varepsilon_1. \end{cases}$$

We require also the function $F_k(x)$ to be continuously differentiable and such that $k^4 + \frac{dF_k}{dx} > 0$. This is possible due to the properties of $f(x)$. Then

$$\max\{\left|F_k(x)\right| : t \in I,\ x \in R\} \leq M_k + \varepsilon_2. \tag{36}$$

Notice that both positive ε_1 and ε_2 can be made arbitrarily small. Denote

$$\Omega_k = \{(t,\, x) :\ 0 \leq t \leq 1, \quad |x(t)| \leq N_k\}.$$

The original problem (23), (24) and the quasi-linear one (35), (24) are equivalent in Ω_k .

The quasi-linear problem (35), (24) can be written in the integral form

$$x(t) = \int_0^1 G_k(t,s) F_k(x(s))\, ds.$$

It follows from (30), (36) that

$$\left|x(t)\right| \leq \Gamma_k \cdot (M_k + \varepsilon_2).$$

If the inequality

$$\Gamma_k \cdot M_k < N_k \tag{37}$$

holds, and ε_2 is such that also $\Gamma_k \cdot (M_k + \varepsilon_2) < N_k$ (and this is the case), then a solution $x(t)$ of the quasi-linear problem (35), (24) satisfies the estimate

$$\big|x(t)\big| < N_k, \quad \forall\, t \in [0,\, 1]$$

and this solution $x(t)$ solves the original problem (23), (24) also. (The oscillatory properties of a solution $x(t)$ depend on the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$.)

It follows from (31), (33) that the inequality (37) takes the form

$$\Gamma_k \cdot \alpha^{\frac{2}{1-p}} \cdot \Big(\frac{k^4}{p}\Big)^{\frac{p}{p-1}} \cdot |p-1| < \Big(\frac{k^4}{\alpha^2}\Big)^{\frac{1}{p-1}} \beta,$$

or

$$\Gamma_k \cdot k^4 < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}. \tag{38}$$

Notice that the inequality (38) is independent of α.
It follows from (30) that the latter inequality can be written in the form

$$k \cdot \frac{\cosh k + (1+\sqrt{2})(e^k + 1)}{2\big|\cosh k \cdot \cos k - 1\big|} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}. \tag{39}$$

Since $\beta > 1$ and $\lim\limits_{p \to 1} \dfrac{p^{\frac{p}{p-1}}}{|p-1|} = +\infty$ the right side of the inequality (39) tends to ∞ as $p \to 1$. Then the inequality (39) holds for arbitrarily large values of k.

If there exist numbers k_j, which belong to the different intervals $\big(\xi_j, \xi_{j+1}\big)$, where ξ_j and ξ_{j+1} are the roots of the equation

$$\cos\xi \cdot \cosh\xi = 1,$$

such that the inequality (37) and/or respectively (39) holds, then there exist different solutions of original problem (23), (24).

To simplify calculations take $k = \pi n$, where $n = 1, 2 \ldots$ In this case the Green's function satisfies the estimates

$$\big|G_k(t,s)\big| \le \Gamma_1(k) = \frac{(1+\sqrt{2})e^k}{k^3(e^k+1)}, \qquad \text{if} \quad k = (2n-1)\pi \tag{40}$$

and

$$\big|G_k(t,s)\big| \le \Gamma_2(k) = \frac{(1+\sqrt{2})e^k}{k^3(e^k-1)}, \qquad \text{if} \quad k = 2n\pi. \tag{41}$$

Therefore, the inequality (37) takes the form

$$k \cdot \frac{(1+\sqrt{2})e^k}{(e^k+1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}, \qquad \text{for} \quad k = (2n-1)\pi, \tag{42}$$

$$k \cdot \frac{(1+\sqrt{2})e^k}{(e^k-1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}, \qquad \text{for} \quad k = 2n\pi. \tag{43}$$

In the table below the results of calculations are provided. It is shown for certain values of k in the form $k = \pi n, \quad n = 1, 2 \ldots$ are good for the inequalities (42) and (43) to be satisfied.

5.3 Remarks on numerically finding solutions

It is seen from the table below that for certain p there exist multiple solutions of the BVP. Basic relations (38) and (39) are independent of the coefficient α^2. However, possible solutions of the BVP, which can be obtained using the quasilinearization process, satisfy the estimate

$$|x_k(t)| \leq \Gamma_k \cdot M_k < N_k = \left(\frac{k^4}{\alpha^2}\right)^{\frac{1}{p-1}} \beta, \tag{44}$$

where the right side depends on α^2.

Set $p = \frac{9}{8}$, for instance. Then at least three solutions $x_\pi(t)$, $x_{2\pi}(t)$ and $x_{3\pi}(t)$ to the BVP are expected to exist (see the table below). Computation shows that $N_\pi = \frac{\pi^{32}}{\alpha^{16}}\beta \approx \frac{1.05359 \cdot 10^{16}}{\alpha^{16}}$. Respectively,

$$N_{2\pi} = \frac{(2\pi)^{32}}{\alpha^{16}}\beta \approx \frac{4.52514 \cdot 10^{25}}{\alpha^{16}}, \quad N_{3\pi} = \frac{(3\pi)^{32}}{\alpha^{16}}\beta \approx \frac{1.95233 \cdot 10^{31}}{\alpha^{16}}.$$

Therefore in order to get reasonably bounded solutions of the BVP one should consider equations with large enough coefficients α^2.

$p = \frac{4}{5}$	$\beta \approx 1.3632$	$k = \pi;\ k = 2\pi$
$p = \frac{5}{6}$	$\beta \approx 1.3553$	$k = \pi;\ k = 2\pi$
$p = \frac{6}{7}$	$\beta \approx 1.3499$	$k = \pi;\ k = 2\pi;\ k = 3\pi$
$p = \frac{7}{8}$	$\beta \approx 1.3461$	$k = \pi;\ k = 2\pi;\ k = 3\pi$
$p = \frac{8}{9}$	$\beta \approx 1.3431$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$
$p = \frac{9}{10}$	$\beta \approx 1.3407$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$
$p = \frac{10}{11}$	$\beta \approx 1.3388$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$
$p = \frac{11}{12}$	$\beta \approx 1.3373$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$
$p = \frac{12}{13}$	$\beta \approx 1.3359$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$
$p = \frac{13}{14}$	$\beta \approx 1.3349$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi;\ k = 6\pi$
...	...	...
$p = \frac{14}{13}$	$\beta \approx 1.3076$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi;\ k = 6\pi$
$p = \frac{13}{12}$	$\beta \approx 1.3065$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$
$p = \frac{12}{11}$	$\beta \approx 1.3053$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi;\ k = 5\pi$
$p = \frac{11}{10}$	$\beta \approx 1.3038$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$
$p = \frac{10}{9}$	$\beta \approx 1.3019$	$k = \pi;\ k = 2\pi;\ k = 3\pi;\ k = 4\pi$
$p = \frac{9}{8}$	$\beta \approx 1.2998$	$k = \pi;\ k = 2\pi;\ k = 3\pi$
$p = \frac{8}{7}$	$\beta \approx 1.2969$	$k = \pi;\ k = 2\pi;\ k = 3\pi$
$p = \frac{7}{6}$	$\beta \approx 1.2933$	$k = \pi;\ k = 2\pi;\ k = 3\pi$
$p = \frac{6}{5}$	$\beta \approx 1.2884$	$k = \pi;\ k = 2\pi$
$p = \frac{5}{4}$	$\beta \approx 1.2813$	$k = \pi;\ k = 2\pi$

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И.Р. Ермаченко, Ф.Ж. Садырбаев. Типы решений в двухточечной нелинейной краевой задаче четвертого порядка: оценки числа решений.

Аннотация. Рассматривается двухточечная краевая задача для нелинейного ОДУ четвертого порядка с монотонной правой частью. Если нелинейное уравнение может быть представлено в квазилинейной форме с нерезонансной линейной частью и оба уравнения эквивалентны в некоторой области D, которой принадлежат решения квазилинейного уравнения, то исходная задача имеет решение. Тогда мы говорим об исходной задаче, что она квазилинеаризуема с данной линейной частью. Мы показываем, что квазилинейная задача имеет решение определенного типа, который

согласуется со свойствами линейной части. Если указанная квазилинеаризация возможна для данного уравнения с существенно различными линейными частями, то исходная краевая задача имеет различные решения.

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I. Jermačenko, F. Sadirbajevs. Ceturtās kārtas divpunktu robežproblēmu atrisinājumu tipi: atrisinājumu skaita novērtējumi.

Anotācija. Tiek pētīta robežproblēma $x^{(4)} = f(t,x)$ (i), $x(0) = x'(0) = 0 = x(1) = x'(1)$ (ii). Pieņemsim, ka vienādojums (i) var būt pierakstīts ekvivalentā formā $x^{(4)} - k^4x = F(t,x)$ (iii) kādā kompaktā (t,x)-apgabalā D (funkcija F ir ierobežota) un kvazilineārās problēmas (iii), (ii) atrisinājumi apmierina nosacījumu $(t,x(t)) \in D$ $\forall t \in [0,1]$. Šajā gadījumā sakām, ka sākotnējā robežproblēma ir kvazilineārizējama ar lineāru daļu $x^{(4)} - k^4x$. Tad parādām, ja sākotnējā robežproblēma var būt kvazilineārizējama attiecībā pret būtiski atšķirīgām lineārām daļām , tad robežproblēmai ir vairāki atrisinājumi. Tiek analizēti ilustratīvie piemēri.

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