Estimations of the number of solutions to some nonlinear second order boundary value problems

S. Ogorodnikova and F. Sadyrbaev

Summary. We consider first two the second order autonomous differential equations with critical points, which allow for detecting an exact number of solutions to the Dirichlet boundary value problem. Then non-autonomous equations with similar behavior of solutions are considered. Estimations from below of the number of solutions to the Dirichlet boundary value problem are given.

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1 Introduction

In the work [1, Ch. 15] estimations of the number of solutions to the boundary value problem

$$x' = h(t, x, y), \quad y' = f(t, x, y) \tag{1.1}$$

$$\begin{array}{rcl} a_1 x(a) - b_1 x'(a) & = & 0, \\ a_2 x(b) - b_2 x'(b) & = & 0 \end{array} \tag{1.2}$$

were obtained. These estimations were based on comparison of the behavior of solutions in some neighborhood of the zero solution and at infinity. Notice that the zero solution exists since $h(t, 0, 0) = f(t, 0, 0) = 0$. It is convenient to explain the result of A. Perov in terms of the angular function $\varphi(t)$, which can be introduced by the relations

$$x = \rho \sin \varphi, \quad y = \rho \cos \varphi, \quad \rho^2 = x^2 + y^2. \tag{1.3}$$

One gets the following equations for the functions φ and ρ :

$$\left\{ \begin{array}{rcl} \varphi' & = & \frac{1}{\rho} \cdot [h \cos \varphi - f \sin \varphi] \\ \rho' & = & h \sin \varphi + f \cos \varphi. \end{array} \right. \tag{1.4}$$

Let φ_0 and φ_1 be the angles which relate respectively to the first and the second of the boundary conditions (1.2).

Set

$$\rho_0 = \sqrt{x^2(a) + y^2(a)}.$$

Suppose that a solution $\varphi(t)$ of the system (1.4), which is defined by the initial condition $\varphi(a) = \varphi_0$ for $\rho_0 \sim 0$, takes exactly m values of the form $\varphi_1(\mathrm{mod}\,\pi)$. Moreover, assume that a solution $\varphi(t)$, which is defined by the initial condition $\varphi(a) = \varphi_0$ and which relates to values $\rho_0 \sim +\infty$, takes n values of the form $\varphi_1(\mathrm{mod}\,\pi)$. Then there exist at least $2|n-m|$ nontrivial solutions of the problem.

The figure 1 visualizes the case of $n = 0$ and $m = 1$. Two possible solutions of the BVP are depicted by bold lines.

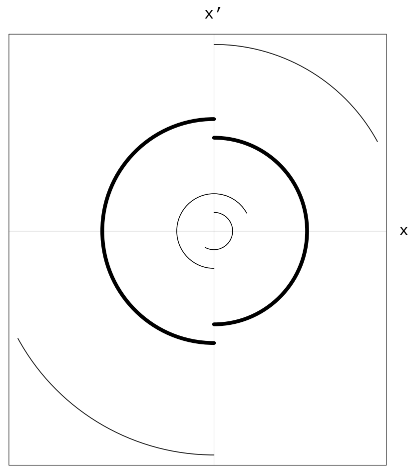

Figure 1: Perov's result ($m = 1,\ n = 0$,), bold - orbits of solutions to BVP; normal - orbits at infinity and at zero.

Due to different rates of whirling of solutions near the zero and at infinity multiple solutions of the problem appear.

The above mentioned result by A. Perov is much more general than that described by Fig. 1, since equations in (1.1) are non-autonomous.

Our aim in this paper is the following. We consider the second order equations, which are equivalent to two-dimensional systems, which are similar to those treated by A. Perov and which, moreover, can have hetero- and homoclinic type solutions.

Our plan is to consider first autonomous equations which have singular points of the type center - saddle. This equation has a homoclinic solution and it may have multiple solutions of the Dirichlet problem.

The results are then generalized to the case of non-autonomous equation, which has a solution, defined on a finite interval and which possesses some properties of a homoclinic solution.

Similar situation is considered for autonomous equations which have singular points of the type saddle - center - saddle. This equation has a heteroclinic solution and it may also have multiple solutions of the Dirichlet problem.

The results were announced in [3].

2 Autonomous equations

Consider the equations

$$\begin{aligned} x'' &= -x + x^3 \\ x'' &= -x + x^2 \end{aligned} \tag{2.1}$$

We will show that the Dirichlet boundary value problems for equations (2.1) have different numbers of solutions.

First consider the problem

$$x'' = -\alpha x + x^2, \tag{2.2}$$

$$x(0) = 0, \quad x(1) = 0, \tag{2.3}$$

where $\alpha > 0$ is a parameter.

The equivalent system

$$x' = y$$

$$y' = -\alpha x + x^2, \tag{2.4}$$

has a center at $(0; 0)$ and a saddle point at $(\alpha; 0)$. The homoclinic orbit connects the saddle point to itself. It has “an infinite” period ([2]).

Proposition 1 *Suppose that the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{2.5}$$

holds, where n *is a non-negative integer. Then the problem (2.2), (2.3) has exactly* $2n+1$ *nontrivial solutions.*

Proof. Consider solutions $x(t; \gamma)$ of equation (2.2), which satisfy the initial conditions

$$x(0) = 0, \quad x'(0) = \gamma. \tag{2.6}$$

Equation (2.4), linearized at zero, takes the form

$$y'' = -\alpha y. \tag{2.7}$$

Solutions of the problem (2.2), (2.6) have exactly n zeros in the interval $(0; 1)$ and does not vanish at $t = 1$ for small $\gamma > 0$. The parameter γ varies from zero value to γ_0, where γ_0 defines a homoclinic orbit. The zeros of solutions $x(t; \gamma)$ monotonically increase and leave the interval $(0; 1]$ passing through the right end point as $\gamma \to \gamma_0$. If for some γ a solution $x(t; \gamma)$ has a zero at $t = 1$, then $x(t; \gamma)$ is a solution to the boundary value problem (2.2), (2.3). Thus n solutions of the problem.

Similarly n solutions to the boundary value problem appear for $\gamma \in (0, -\gamma_0)$.

Solutions of the problem (2.2), (2.6) do not vanish for $\gamma > \gamma_0$.

Solutions $x(t; \gamma)$ of the problem (2.2), (2.6) do not vanish also for $-\gamma_0 - \varepsilon\gamma < -\gamma_0$, where $\varepsilon > 0$ is sufficiently small. If γ continues to change to $-\infty$, for γ large enough in modulus an extra zero of a solution $x(t; \gamma)$ appears, which belongs to the interval $(0; 1]$. This means that there exists some $\gamma_1 \in (-\infty, -\gamma_0)$ such that a solution $x(t; \gamma_1)$ solves the problem (2.2), (2.3).

Hence, we considered 4 regions on a phase plane: two of them inside the homoclinic orbit, (we discovered $2n$ solutions of the problem (2.2), (2.3) there) and another two regions outside the homoclinic orbit. The positive outer region ($\gamma > 0$) does not contain a solution. The negative outer region ($\gamma < 0$) yields exactly one solution to the problem. Therefore the problem (2.2), (2.3) has exactly $2n+1$ solutions. □

Consider the problem

$$x'' = -\alpha x + x^3, \tag{2.8}$$

(2.3), where $\alpha > 0$. The equivalent system

$$\begin{aligned} x' &= y, \\ y' &= -\alpha x + x^3 \end{aligned} \tag{2.9}$$

has a center at $(0;0)$ and two saddle points at $(-\sqrt{\alpha};0)$ and $(\sqrt{\alpha};0)$. The heteroclinic orbit connects two saddle points. The respective heteroclinic solution has "an infinite" period ([2]).

Proposition 2 *Let the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{2.10}$$

hold, where n is a non-negative integer. Then the problem (2.8), (2.3) has exactly $2n$ nontrivial solutions.

Proof. Consider solutions $x(t;\gamma)$ of the initial value problem (2.8), (2.6). Linearization of (2.8) at zero yields equation (2.7).

Solutions $x(t;\gamma)$ of the problem (2.8), (2.6) have exactly n zeros in the interval $(0;1)$ and do not vanish at $t = 1$, if $\gamma > 0$ is small enough. The zeros of $x(t;\gamma)$ increase monotonically and pass through the point $t = 1$ as γ increases from 0 to γ_0, where γ_0 relates to heteroclinic orbit. Any γ such that a solution $x(t;\gamma)$ has a zero at $t = 1$, gives rise to a solution of the boundary value problem. Hence n solutions of the problem.

Similarly exactly n solutions to the problem appear for $\gamma \in (\gamma_0, 0)$.

Solutions $x(t;\gamma)$ of the initial value problem (2.8), (2.6) do not vanish for $|\gamma| > |\gamma_0|$.

Therefore totally the number of nontrivial solutions of the problem (2.8), (2.3) is $2n$. □

3 Non-autonomous equations

Consider the problem

$$x'' = f(t,x), \tag{3.1}$$

$$x(0) = x(1) = 0, \tag{3.2}$$

where function f satisfies the conditions:

(A1) f and f_x are $C(I \times R)$-functions;

(A2) $f(t,0) \equiv 0$;

(A3) $f(t,x) > c|x|^p$ for $t \in I, |x| > M$, where $c > 0, p > 1, M > 0$ are constants;

(A4) there exists a solution $\eta(t)$ of the problem (3.1), $\eta(0) = 0, \eta'(0) > 0$ such that $\eta(t)$ does not vanish in the interval $(0; 1]$;

(A5) there exists a solution $\xi(t)$ of the problem (3.1), $\xi(0) = 0, \xi'(0) < 0$ such that $\xi(t)$ does not vanish in the interval $(0; 1]$;

(A6) solutions of equation (3.1) extend to the interval $(0; 1]$.

Theorem 3.1 *Let the conditions **(A1) - (A5)** hold. Suppose that a solution $y(t)$ of the Cauchy problem*

$$y'' = f_x(t, 0)\, y, \tag{3.3}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{3.4}$$

has exactly n zeros in the interval $(0, 1)$ and $y(1) \neq 0$.

Then the problem has at least $2n + 1$ nontrivial solutions.

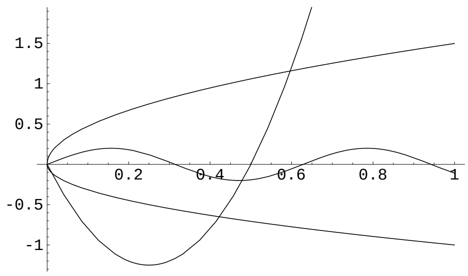

Figure 2: Visualization of Theorem 3.1.

We need some auxiliary results before passing to prove the theorem.

Lemma 3.1 *Solutions $x(t; \beta)$ of the initial value problem*

$$x'' = c|x|^p, \quad x(0) = -M, \quad x'(0) = -\beta, \quad \beta > 0 \tag{3.5}$$

for β large enough have a minimum at some point $T(\beta)$ and $T(\beta) \to +\infty$ as $\beta \to +\infty$.

Proof. In order to avoid difficulties manipulating with negative numbers we consider a symmetric problem

$$y'' = -cy^p, \quad y(0) = M, \quad y'(0) = \beta, \quad \beta > 0, \tag{3.6}$$

looking for solutions with a point of maximum. Multiplying both sides of the equation in (3.6) by $2y'$ and integrating, one gets

$$y'^2(t) - y'^2(0) = -\frac{2c}{p+1}\left(y^{p+1}(t) - y^{p+1}(0)\right)$$

or

$$y'^2(t) - \beta^2 = -\frac{2c}{p+1}\left(y^{p+1}(t) - M^{p+1}\right).$$

At a point of maximum $T > 0$ the relations $y'^2(T) = 0$ and, therefore,

$$y_{\max} = y(T) = \left[\frac{p+1}{2c}\beta^2 + M^{p+1}\right]^{\frac{1}{p+1}} \tag{3.7}$$

hold. Then

$$y'^2(t) = -\frac{2c}{p+1}\left(y^{p+1}(t) - M^{p+1}\right) + \beta^2$$

and

$$\frac{dy}{dt} = \sqrt{\beta^2 - \frac{2c}{p+1}\,[y^{p+1} - M^{p+1}]},$$

$$\frac{dy}{\sqrt{\beta^2 + \frac{2c}{p+1}M^{p+1} - \frac{2c}{p+1}y^{p+1}}} = dt,$$

$$\int_0^{y_{max}} \frac{dy}{\sqrt{\frac{2c}{p+1}(-M)^{p+1} + \beta^2 - \frac{2c}{p+1}y^{p+1}}} = \int_0^T dt = T$$

$$\int_0^{y_{max}} \frac{dy}{\sqrt{\frac{2cM^{p+1}+(p+1)\beta^2}{p+1}}} \cdot \sqrt{1 - \frac{2c}{2cM^{p+1}+(p+1)\beta^2} \cdot y^{p+1}} =$$

$$= \sqrt{\frac{p+1}{2cM^{p+1}+(p+1)\beta^2}} \cdot \int_0^{y_{max}} \frac{dy}{\sqrt{1 - \frac{2c}{2cM^{p+1}+(p+1)\beta^2} \cdot y^{p+1}}} =$$

$$\left|\begin{array}{l} \frac{2c}{2cM^{p+1}+(p+1)\beta^2} \cdot y^{p+1} = \xi^{p+1} \\ \xi = \sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}}\, y \\ d\xi = \sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}}\, dy \end{array}\right|$$

$$= \frac{(p+1)^{\frac{1}{2}}}{(2cM^{p+1}+(p+1)\beta^2)^{\frac{1}{2}}} \cdot \int_0^1 \frac{d\xi}{\sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}} \cdot \sqrt{1-\xi^{p+1}}} =$$

$$= \frac{(p+1)^{\frac{1}{2}}}{(2cM^{p+1}+(p+1)\beta^2)^{\frac{1}{2}}} \cdot \frac{(2cM^{p+1}+(p+1)\beta^2)^{\frac{1}{p+1}}}{(2c)^{\frac{1}{p+1}}} \cdot \int_0^1 \frac{d\xi}{\sqrt{1-\xi^{p+1}}} =$$

$$= \frac{(p+1)^{\frac{1}{2}}}{(2c)^{\frac{1}{p+1}}} \cdot \frac{1}{[2cM^{p+1}+(p+1)\beta^2]^{\frac{p}{2(p+1)}}} \cdot \int_0^1 \frac{d\xi}{\sqrt{1-\xi^{p+1}}}.$$

Since the integral $\displaystyle\int_0^1 \frac{d\xi}{\sqrt{1-\xi^{p+1}}}$ is finite, the expression in the last line tends to zero as β goes to infinity. □

Corollary 3.1 *For any $t_1 > 0$ there exists a solution $x(t;\beta)$ of the problem*

$$x'' = cx^p, \quad x(t_1) = -M, \quad x'(t_1) = -\beta, \quad \beta > 0, \tag{3.8}$$

such that it has a minimum point at $t_1 + T(\beta)$ and $x(t_1 + 2T) = -M$, where $T(\beta) \to 0$ as $\beta \to +\infty$.

Proof. Since equation in (3.8) is autonomous, any function $x(t + const)$ is a solution if $x(t)$ is. The existence of a point of minimum $T(\beta)$ follows from Lemma 3.1. Notice also that the function $z(t) = x(t_1 + 2T - t)$ is a solution of the equation. By combining $x(t)$, defined in the interval $[t_1, t_1 + T]$ and symmetric solution $z(t)$, defined in $[t_1 + T, t_1 + 2T]$, a solution $x(t;\beta)$ can be obtained. □

Lemma 3.2 *Suppose the conditions **(A1)** and **(A3)** hold. Then solution $x(t;\alpha)$ of the Cauchy problem*

$$x'' = f(t,x), \quad x(0) = 0, \quad x'(0) = \alpha \tag{3.9}$$

vanishes at some point $t_1(\alpha) \in (0,1)$ for α negative valued and large enough in modulus.

Proof. Equation $y'' = cy^p$ written in polar coordinates $y = \varrho\sin\Phi, \quad y' = \varrho\cos\Phi$ looks like

$$\begin{cases} \Phi' &= \quad \cos^2\Phi - c\varrho^{p-1}|\sin\Phi|\sin\Phi, \\ \varrho' &= \varrho\sin\Phi\cos\Phi + c\varrho^p|\sin\Phi|\cos\varphi. \end{cases} \tag{3.10}$$

Equation $x'' = f(t,x)$, where by the condition (A3) $f(t,x(t)) = cx^p(t) + \varepsilon(t)$ ($\varepsilon(t) \geq 0$) for any solution $x(t)$ such that $x(t) < -M$, for respective t takes the form

$$\begin{cases} \varphi' &= \quad \cos^2\varphi - c\rho^{p-1}|\sin\varphi|\sin\varphi + \frac{\varepsilon(t)}{\rho(t)}\sin\varphi, \\ \rho' &= \rho\sin\varphi\cos\varphi + c\rho^p|\sin\varphi|\cos\varphi + \varepsilon(t)\cos\varphi, \end{cases} \tag{3.11}$$

where

$$x = \rho\sin\varphi, \quad x' = \rho\cos\varphi. \tag{3.12}$$

We are interested in negative solutions $y(t)$ and $x(t)$. Therefore angular functions $\Phi(t)$ and $\varphi(t)$ can be considered in the interval $(\pi, 2\pi)$ and, therefore, all the addends in the right sides of (3.10) and (3.11) are non-negative.

The assertion of Lemma can be proved by comparison of the angular functions $\Phi(t)$ and $\varphi(t)$. Alternatively the technique of the work [4] (Lemmas 4.1 and 4.2) can be applied to prove Lemma 3.2.

Proof of the theorem. We will use the polar coordinates introduced by (3.12). Notice that $\rho(t)$ cannot vanish for nontrivial solutions since $x \equiv 0$ is a solution. Equation (3.1) in polar coordinates takes the form

$$\begin{cases} \varphi' &= \quad \cos^2\varphi - \frac{1}{\rho}\cdot f(t,\rho\sin\varphi)\cdot\sin\varphi \\ \rho' &= \rho\sin\varphi\cos\varphi + f(t,\rho\sin\varphi)\cdot\cos\varphi \end{cases}$$

The linear equation of variations (3.3), converted into polar coordinates using the formulas

$$y = r\sin\Theta, \quad y' = r\cos\Theta,$$

turns to

$$\begin{cases} \Theta' &= \quad \cos^2\Theta - \frac{1}{r}\cdot f(t, r\sin\Theta)\cdot\sin\varphi \\ r' &= r\sin\Theta\cos\Theta + f(t, r\sin\Theta)\cdot\cos\Theta \end{cases}$$

Consider solutions $x(t,\alpha)$ of the initial value problem (3.9). For $\alpha \sim 0$ solutions $x(t,\alpha)$ stay in a small neighborhood of the trivial solution. Functions $\rho(t;\alpha)$ and $\varphi(t;\alpha)$ tend uniformly in $t \in [0,1]$ respectively to $r(t)$ and $\Theta(t)$ as $\alpha \to 0$. It follows from the conditions of the theorem that $x(t,\alpha)$ has exactly n zeros in $(0,1)$ and $x(1;\alpha) \neq 0$ if α is small in modulus.

Consider now several cases.

Case 1. $\alpha \in (0, \eta'(0))$. One has that $\varphi(1;\alpha) \in (\frac{\pi}{2}n, \frac{\pi}{2}(n+1))$ for small values of α. Let α_η relates to a solution $\eta(t)$. This solution by the condition (A4) does not vanish in $(0,1]$.

Then the function $\varphi(t;\alpha_\eta)$ does not take values of the form $\frac{\pi}{2}(\mod \pi)$. Since $\varphi(t;\alpha)$ continuously depends on α there exist values α_i $(i = 1, \ldots, n)$ such that $\varphi(1;\alpha_i) = \frac{\pi}{2}i$. Hence at least n solutions of the BVP.

Case 2. Similarly at least n solutions of the problem (3.1), (3.2) for $\alpha \in (\xi'(0), 0)$.

Case 3. Let $\alpha \in (-\infty, \xi'(0))$. Since solutions of the initial value problem (3.9) for $\alpha \in (\xi'(0) - \varepsilon, \xi'(0))$ do not vanish for $\varepsilon > 0$ small enough, the respective values of the angular function $\varphi(1;\alpha) \in (\pi, 2\pi)$. Since for some solution of the IVP (3.9) there exists a zero in the interval $(0, 1)$, the function $\varphi(1, \alpha)$ attains values greater than 2π for some $\alpha \in (-\infty, \xi'(0))$. Then there must exist a value $\alpha_0 \in (-\infty, \xi'(0))$ such that $\varphi(1;\alpha_0) = 2\pi$. Hence a solution to the boundary value problem.

Summing up, one concludes that the BVP under consideration has at least $2n + 1$ nontrivial solutions. □

Remark 3.1 *The assumption (A6) is technical and the results are true also if non-extendable solutions are allowed.*

Remark 3.2 *Since the second autonomous equation in (ref2 1) meets all the conditions of Theorem 3.1 the above estimate of the number of solutions is sharp.*

Consider the problem (3.1), (3.2), where f satisfies the conditions $(A1)$, $(A2)$, $(A4)$, $(A5)$, $(A6)$ and $(A3)$ is replaced by (B).

(B) $xf(t, x) > 0$ and $|f(t, x)| > c|x|^p$ for $t \in I, |x| > M$, where $c > 0, p > 1, M > 0$ are constants;

Theorem 3.2 *Let the conditions (A1), (A2), (B), (A4), (A5) hold. Assume also that solutions $y(t)$ of the Cauchy problem*

$$y'' = f_x(t, 0)y \tag{3.13}$$

$$y(0) = 0, \quad y'(0) = 1. \tag{3.14}$$

have exactly n zeros in the interval $(0, 1)$ and $y(1) \neq 0$.

Then the problem has at least $2n$ nontrivial solutions.

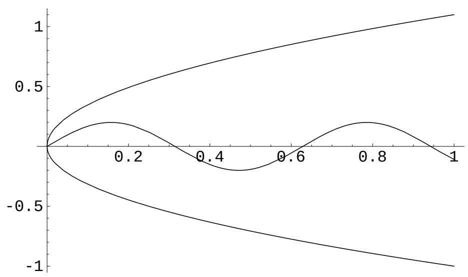

Figure 3: Visualization of Theorem 3.2.

The proof is similar to that of Theorem 3.1 and we omit it. The remarks analogous to Remark 3.1 and Remark 3.2 are true.

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С. Огородникова, Ф.Ж. Садырбаев. Оценки числа решений некоторых нелинейных краевых задач второго порядка.

Аннотация. В начале рассматриваются две краевые задачи с условиями Дирихле для автономных уравнений с критическими точками, допускающими оценку числа решений. Затем изучаются неавтономные уравнения со сходным поведением решений. Даются оценки числа решений задачи Дирихле.

УДК 517.51 + 517.91

S. Ogorodnikova, F. Sadirbajevs. Otrās kārtas nelineāro robežproblēmu atrisinājumu skaita novērtējumi.

Anotācija. Sākumā divas nelineāras robežproblēmas tiek apskatītas otrās kārtas diferenciālvienādojumiem ar kritiskiem (singulariem) punktiem. Iegūtie rezultāti tiek pārnesti uz neautonomiem diferenciālvienādojumiem . Noformulētas divas teorēmas par robežproblēmu atrisinājumu skaitu.

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