Explicit solutions of non-autonomous Emden – Fowler type equations

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Summary. We consider the nonlinear equations of the form

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad ' = \frac{d}{dt}, \quad \varepsilon > 0, \quad q \in C(R, (0, +\infty)), \tag{1}$$

where $q(t)$ are some specific t-dependent functions such that explicit formulas for solutions can be given. We study properties of solutions, construct principal and non-principal (in the sense of Hartman) solutions and compute the characteristic numbers introduced by Nehari.

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1 Introduction

We consider the Emden – Fowler equation

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0, \tag{2}$$

where $q(t)$ is a positive valued continuous function. The theory of such equations is rich. The interested reader may consult the book [10] for some basic facts. Any nontrivial solution x of (2) satisfies the inequality $xx'' < 0$ if $x \neq 0$. This means that solutions are convex towards the outside. Equations of the type (2) are known to have oscillatory solutions (that is, solutions with infinitely many zeros), nonoscillatory solutions (that is, solutions with at most finite number of zeros), solutions tending to a linear function at infinity (including constants), singular solutions of superlinear character (that is, solutions which behave on a finite interval like the function $t \sin \frac{1}{t}$), singular solutions of sublinear character (that is solutions, which smoothly "enter" the trivial solution etc. So, despite of its relatively simple structure, the Emden – Fowler equation (2) contains a lot of examples of different kind solutions.

Some regularity is brought to the theory of the Emden – Fowler superlinear equation by the results of Nehari ([2], [3]), which are variational in nature. Brief description of the Nehari theory is given in the fourth section of this work. Other sections deal with basic formulas for solutions of both the initial value problem and boundary value problem (Section 2), examples of the so called principal and non-principal solutions in the sense of Hartman (Section 3), the characteristic numbers $\lambda_n(a,b)$, introduced by Nehari and called in this text by the Nehari numbers (Section 4).

2 Formulas for solutions

2.1 Initial value problem

Consider the initial value problem

$$x'' = -\frac{k}{(pt+q)^{2\varepsilon+4}}|x|^{2\varepsilon}x, \quad x(a)=0, \quad x'(a)=\beta, \quad k>0, \tag{3}$$

where $\xi(t) := pt+q > 0$ on the interval of definition.

Let $S(t)$ stand for a solution of the Cauchy problem

$$u'' = -(\varepsilon+1)|u|^{2\varepsilon}u, \quad u(0)=0, \quad u'(0)=1. \tag{4}$$

Notice that $S(t)$ is periodic with period depending on ε. The following statement generalizes the respective result in [13].

Proposition 2.1 *Solutions $x(t;\beta)$ of the problem (3), where $\beta \geq 0$, are given by the formula*

$$x(t) = \beta^{\frac{1}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pt+q) S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right), \tag{5}$$

where

$$r = \frac{(pa+q)^2(\varepsilon+1)}{k}. \tag{6}$$

Proof. A solution $u(t)$ of the problem

$$u'' = -\widetilde{k}|u|^{2\varepsilon}u, \quad u(a)=0, \quad u'(a)=\beta \geq 0,$$

where $\widetilde{k} > 0$, is given by

$$u(t) = \beta^{\frac{1}{\varepsilon+1}} \widetilde{k}^{-\frac{1}{2\varepsilon+2}} (\varepsilon+1)^{\frac{1}{2\varepsilon+2}} S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} \widetilde{k}^{\frac{1}{2\varepsilon+2}} (\varepsilon+1)^{-\frac{1}{2\varepsilon+2}} (t-a)\right),$$

where $S(t)$ is described above.

We are looking for a solution of (3) in the form

$$x(t) = \xi(t) u\big(w(t)\big),$$

where $\xi(t) > 0$ for $t \in [a,b]$. Then

$$\frac{dx}{dt} = \frac{d\xi}{dt} u\big(w(t)\big) + \xi(t) \frac{du}{dw}\frac{dw}{dt},$$

$$\frac{d^2x}{dt^2} = \frac{d^2\xi}{dt^2}u\big(w(t)\big) + \left(2\frac{d\xi}{dt}\frac{dw}{dt} + \xi\frac{d^2w}{dt^2}\right)\frac{du}{dw} + \xi(t)\left(\frac{dw}{dt}\right)^2\frac{d^2u}{dw^2}.$$

Suppose that $\xi(t)$ and $w(t)$ are such that

$$\xi'' = 0, \qquad 2\xi'w' + \xi w'' = 0$$

and $w(a) = a,\ w'(a) = \frac{1}{\xi(a)}$. Evidently $\xi(t) = pt + q$.

One has that:

$$x(a) = \xi(a)u\big(w(a)\big) = \xi(a)u(a) = 0,$$

$$x'(a) = \xi(a)u\big(w(a)\big) + \xi(a)u'\big(w(a)\big)w'(a) = \xi(a)u'(a)\frac{1}{\xi(a)} = \beta.$$

We wish to define a function $w(t)$ now. Consider two cases.

A) Case $\xi' \neq 0$, or $p \neq 0$. One has subsequently that

$$2\xi'w' + \xi w'' = 0, \quad 2pw' + (pt+q)w'' = 0, \quad 2pw + \int (pt+q)w''dt = c_1,$$

$$\int (pt+q)w''dt = \int udv = \begin{vmatrix} u = pt+q \\ du = pdt \\ dv = w''dt \\ v = w' \end{vmatrix} = uv - \int vdu$$
$$= (pt+q)w' - p\int w'dt = (pt+q)w' - pw,$$

$$2pw + (pt+q)w' - pw = c_1, \quad (pt+q)w' = c_1 - pw, \quad (pt+q)\frac{dw}{dt} = c_1 - pw,$$

$$\frac{dw}{c_1 - pw} = \frac{dt}{pt+q}, \quad -\frac{1}{p}\frac{d(c_1 - pw)}{c_1 - pw} = \frac{1}{p}\frac{d(pt+q)}{pt+q}$$

$$-\ln(c_1 - pw) = \ln(pt+q) + \ln c_2, \quad \ln(c_1 - pw) = -\ln c_2(pt+q)$$

$$c_1 - pw = \frac{1}{c_2(pt+q)}, \quad w(t) = \frac{c_1}{p} - \frac{1}{c_2 p\xi(t)}$$

$$w'(t) = -\frac{1}{c_2 p}\left(\frac{1}{\xi(t)}\right)' = -\frac{1}{c_2 p}\left(-\frac{\xi'(t)}{\xi^2(t)}\right) = \frac{p}{c_2 p\xi^2(t)} = \frac{1}{c_2\xi^2(t)}$$

Taking into account that $w'(a) = \frac{1}{\xi(a)}$ one obtains $\frac{1}{\xi(a)} = \frac{1}{c_2\xi^2(a)}$ and $c_2 = \frac{1}{\xi(a)}$. Notice that $w(a) = a$, then

$$a = \frac{c_1}{p} - \frac{1}{\frac{1}{\xi(a)}p\,\xi(a)}, \quad c_1 = 1 + ap.$$

One obtains

$$w(t) = \frac{1+pa}{a} - \frac{1}{\frac{1}{\xi(a)}p\xi(t)} = a + \frac{1}{p} - \frac{\xi(a)}{p\xi(t)}$$
$$= a + \frac{\xi(t) - \xi(a)}{p\xi(t)} = a + \frac{pt + q - (pa+q)}{pt+q} = a + \frac{p(t-a)}{p(pt+q)} = a + \frac{t-a}{pt+q}$$

and, finally,

$$w(t) = a + \frac{t-a}{pt+q}. \tag{7}$$

B) Case $\xi' = p = 0$. Notice that $q \neq 0$. Then $w'' = 0$ and $w(t) = c_1 t + c_2,\ w'(t) = c_1$. It follows from $w'(a) = \frac{1}{\xi(a)} = \frac{1}{q}$ that $c_1 = \frac{1}{q}$. One has from $w(a) = a$ that

$$a = \frac{a}{q} + c_2, \quad c_2 = a - \frac{a}{q}.$$

At the end $w(t) = \frac{t}{q} + a - \frac{a}{q}$ and

$$w(t) = a + \frac{t-a}{q}. \tag{8}$$

Since (7) contains (8), in any case, either $\xi' \neq 0$ or $\xi' = 0$, the relation (7) holds.

It follows from

$$x'' = \xi w'^2 u'' = \xi w'^2(-\widetilde{k}u^{2\varepsilon+1}) = -\widetilde{k}\xi w'^2 \frac{x^{2\varepsilon+1}}{\xi^{2\varepsilon+1}} = -\widetilde{k}\frac{w'^2}{\xi^{2\varepsilon}}x^{2\varepsilon+1} = -q(t)x^{2\varepsilon+1},$$

that

$$q(t) = \widetilde{k}\frac{w'^2}{\xi^{2\varepsilon}}, \quad w(t) = a + \frac{t-a}{pt+q}, \quad w'(t) = \frac{pt+q-(t-a)p}{(pt+q)^2},$$

$$w'(t) = \frac{pa+q}{(pt+q)^2}, \quad w'^2(t) = \frac{(pa+q)^2}{(pt+q)^4} = \frac{\xi^2(a)}{\xi^4(t)}, \quad q(t) = \frac{\widetilde{k}\xi^2(a)}{\xi^{2\varepsilon+4}(t)}.$$

Therefore a solution $x(t)$ of the problem

$$x'' = -\frac{\widetilde{k}\xi^2(a)}{\xi^{2\varepsilon+4}(t)}x^{2\varepsilon}x, \quad x(a) = 0, \quad x'(a) = \beta \geq 0$$

is given by

$$x(t) = \xi(t)u\big(w(t)\big)$$

or

$$x(t) = \beta^{\frac{1}{\varepsilon+1}}\widetilde{k}^{-\frac{1}{2\varepsilon+2}}(\varepsilon+1)^{\frac{1}{2\varepsilon+2}}(pt+q)S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}}\widetilde{k}^{\frac{1}{2\varepsilon+2}}(\varepsilon+1)^{-\frac{1}{2\varepsilon+2}}\left(a + \frac{t-a}{pt+q} - a\right)\right)$$

or

$$x(t) = \beta^{\frac{1}{\varepsilon+1}}\widetilde{k}^{-\frac{1}{2\varepsilon+2}}(\varepsilon+1)^{\frac{1}{2\varepsilon+2}}(pt+q)S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}}\widetilde{k}^{\frac{1}{2\varepsilon+2}}(\varepsilon+1)^{-\frac{1}{2\varepsilon+2}}\frac{t-a}{pt+q}\right).$$

Choose

$$\widetilde{k} = \frac{k}{\xi^2(a)}.$$

Then a solution $x(t)$ of the problem

$$x'' = -\frac{k}{(pt+q)^{2\varepsilon+4}}x^{2\varepsilon+1}, \quad x(a) = 0, \quad x'(a) = \beta$$

is given by

$$x(t) = \beta^{\frac{1}{\varepsilon+1}} \left(\frac{k}{(pa+q)^2}\right)^{-\frac{1}{2\varepsilon+2}} (\varepsilon+1)^{\frac{1}{2\varepsilon+2}} (pt+q) \cdot$$
$$\cdot S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} \left(\frac{k}{(pa+q)^2}\right)^{\frac{1}{2\varepsilon+2}} (\varepsilon+1)^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right)$$

or

$$x(t) = \beta^{\frac{1}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pt+q) S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right),$$

where

$$r = \frac{(pa+q)^2(\varepsilon+1)}{k}. \blacktriangleleft$$

Remark 2.1. Solutions of the problem (3) for $\beta < 0$ can be obtained from the formula (5) also since for any solution $x(t)$ of the equation (2) the function $-x(t)$ also is a solution.

Corollary 2.1 *Suppose that the function $pt+q$ in (3) is increasing for $t > a$ (and coefficient $q(t)$ is, respectively, decreasing). Then intervals I_i between two consecutive zeros t_{i-1} and t_i (if any) of a solution $x(t;\beta)$ increase, as well as absolute values of magnitudes of $x(t;\beta)$ in I_i.*

On the other hand, if the function $pt+q$ in (3) is decreasing for $t \in [a,b]$ (and coefficient $q(t)$ increasing). Then intervals I_i between two consecutive zeros of a solution $x(t;\beta)$ decrease, as well as absolute values of magnitudes of $x(t;\beta)$ in I_i.

If $p = 0$ (the function $pt+q$ reduces to a constant), then $x(t;\beta)$ is periodic with the half period $T = \frac{2Aq}{\rho}$, where

$$A = \int_0^1 \frac{du}{\sqrt{1-u^{2\varepsilon+2}}}, \quad \rho = |\beta|^{\frac{\varepsilon}{\varepsilon+1}} \cdot r^{-\frac{1}{2\varepsilon+2}}. \tag{9}$$

Proof. Follows immediately from (5).

Remark 2.2. The assertion above follows also from the general theory of the Emden – Fowler equation (see, for example, [8]).

Corollary 2.2 *The zeros t_i (if any) of a solution $x(t;\beta)$ are monotonically decreasing functions of $\beta > 0$.*

Proof. The zeros t_i of $x(t;\beta)$ can be found by analysis of a function $S\left(\rho\frac{t-a}{pt+q}\right)$. The zeros t_i are to be found from the relations

$$\rho\frac{t_i - a}{pt_i + q} = 2Ai, \quad i = 1, 2, \dots,$$

where A and ρ are as above. Notice that $\tau_i := 2Ai$ are zeros of $S(t)$.

Computation gives

$$t_i = a + 2Ai\frac{pa+q}{\rho - 2Aip}. \qquad \blacktriangleleft \tag{10}$$

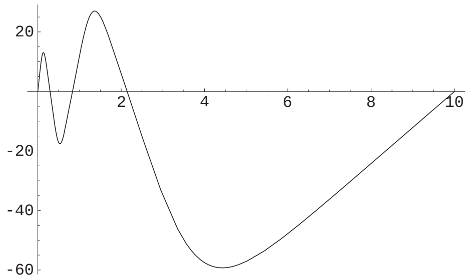


Figure 1: Solution of $x'' = -\frac{2}{(t+1)^6}|x|^2 x, \quad x(0) = 0, \quad x'(0) = 133.1.$

2.2 Boundary value problem

Consider the problem

$$x'' = -\frac{k}{(pt+q)^{2\varepsilon+4}}|x|^{2\varepsilon}x, \quad x(a) = 0, \quad x(b) = 0, \tag{11}$$

where $\xi(t) = pt + q > 0$ in the interval $[a; b]$.

The statement below follows from the results of precedent subsection.

Proposition 2.2 *The problem (11) has (up to multiplication by -1) infinitely many solutions, which are given by the formula*

$$x(t) = \beta^{\frac{1}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pt+q) S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right). \tag{12}$$

For any positive integer n there exists a solution $x_n(t)$ of the form (12) *with*

$$\beta = r^{\frac{1}{2\varepsilon}} \left(\frac{pb+q}{b-a}\right)^{\frac{\varepsilon+1}{\varepsilon}} (2An)^{\frac{\varepsilon+1}{\varepsilon}}, \tag{13}$$

which has exactly $n-1$ zeros in (a, b):

$$t_0 = a < t_1 < t_2 < \cdots < t_n = b,$$

$$t_i = \frac{na\frac{pb+q}{b-a} + iq}{n\frac{pb+q}{b-a} - ip} \quad (i = 0, 1, \ldots, n).$$

2.3 Non-monotone coefficients

Consider the problem

$$x'' = -\frac{k}{\big(\xi(t)\big)^{2\varepsilon+4}}|x|^{2\varepsilon}x, \quad x(a) = 0,\, x(b) = 0, \tag{14}$$

where $\xi(t)$ is a piece-wise linear function composed of two linear segments.

Let $0 \le a < c < b$. Consider the functions

$$\begin{gathered} \xi_1(t) = p_1 t + q_1 > 0, \quad a \le t \le c, \\ \xi_2(t) = p_2 t + q_2 > 0, \quad c \le t \le b, \\ p_1 c + q_1 = \xi_1(c) = \eta_1 = \xi_2(c) = p_2 c + q_2, \\ \eta_0 = \xi_1(a) = p_1 a + q_1, \quad \eta_2 = \xi_2(b) = p_2 b + q_2. \end{gathered}$$

The respective problem is

$$x'' = -q(t)\,|x|^{2\varepsilon}x, \quad x(a) = 0, \quad x(b) = 0, \tag{15}$$

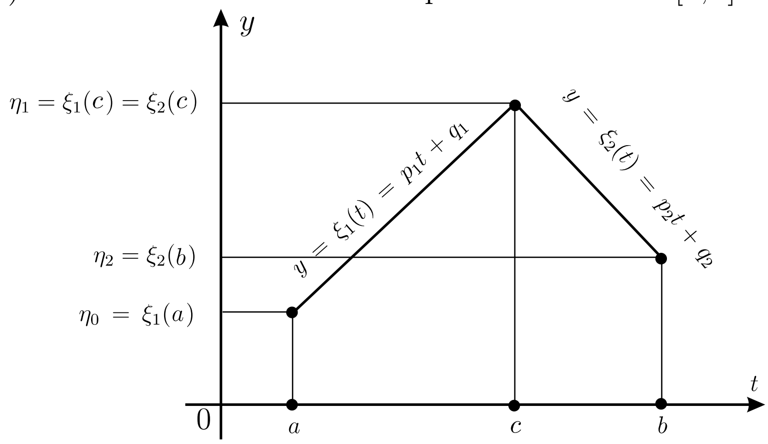
where

$$q(t) = \begin{cases} q_1(t), & \text{for} \quad a \le t \le c, \\ q_2(t), & \text{for} \quad c \le t \le b, \end{cases}$$

$$\begin{aligned} q_1(t) &= \frac{k}{\big(\xi_1(t)\big)^{2\varepsilon+4}} = \frac{k}{(p_1 t + q_1)^{2\varepsilon+4}}, \quad a \le t \le c, \\ q_2(t) &= \frac{k}{\big(\xi_2(t)\big)^{2\varepsilon+4}} = \frac{k}{(p_2 t + q_2)^{2\varepsilon+4}}, \quad c \le t \le b, \end{aligned}$$

the coefficient $k > 0$.

The function $q(t)$ so defined is continuous and positive valued in $[a; b]$.



Consider the problem (15), where function $q(t)$ is continuous and positive valued. We are looking for solutions $x(t)$ with the property that

$$x(c) = 0, \quad x'(c) = 0. \tag{16}$$

The problem (15) decomposes as follows:

$$(\mathbf{A_1}): \quad x_1'' = -\frac{k}{(p_1 t + q_1)^{2\varepsilon+4}}\, x_1^{2\varepsilon+1}, \quad x_1(a) = 0, \quad x_1(c) = 0, \tag{17}$$

$$(\mathbf{A_2}): \quad x_2'' = -\frac{k}{(p_2 t + q_2)^{2\varepsilon+4}}\, x_2^{2\varepsilon+1}, \quad x_2(c) = 0, \quad x_2(b) = 0. \tag{18}$$

We wish solutions $x_1(t)$ and $x_2(t)$ to satisfy the smoothness condition (16).

Let n_1 and n_2 be two positive integers.

A solution of ($\mathbf{A_1}$) has the form:

$$x_1(t) = \operatorname{sign}(\beta_1)|\beta_1|^{\frac{1}{\varepsilon+1}} r_1^{\frac{1}{2\varepsilon+2}} (p_1 t + q_1) S\left(|\beta_1|^{\frac{\varepsilon}{\varepsilon+1}} r_1^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{p_1 t + q_1}\right),$$

where

$$r_1 = \frac{(p_1 a + q_1)^2(\varepsilon+1)}{k}, \quad |\beta_1| = r_1^{\frac{1}{2\varepsilon}} \left(\frac{p_1 c + q_1}{c-a}\right)^{\frac{\varepsilon+1}{\varepsilon}} (2An_1)^{\frac{\varepsilon+1}{\varepsilon}}.$$

Function $x_1(t)$ has exactly (n_1-1) zeros in the open interval $(a;c)$. The derivative is given by

$$x_1'(t) = \operatorname{sign}(\beta_1)|\beta_1|^{\frac{1}{\varepsilon+1}} r_1^{\frac{1}{2\varepsilon+2}}\, p_1 S\left(|\beta_1|^{\frac{\varepsilon}{\varepsilon+1}} r_1^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{p_1 t + q_1}\right) + \\ + \beta_1 S'\left(|\beta_1|^{\frac{\varepsilon}{\varepsilon+1}} r_1^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{p_1 t + q_1}\right) \frac{p_1 a + q_1}{p_1 t + q_1}.$$

A solution of the problem ($\mathbf{A_2}$) has the form:

$$x_2(t) = \operatorname{sign}(\beta_2)|\beta_2|^{\frac{1}{\varepsilon+1}} r_2^{\frac{1}{2\varepsilon+2}} (p_2 t + q_2) S\left(|\beta_2|^{\frac{\varepsilon}{\varepsilon+1}} r_2^{-\frac{1}{2\varepsilon+2}} \frac{t-c}{p_2 t + q_2}\right),$$

where

$$r_2 = \frac{(p_2 c + q_2)^2(\varepsilon+1)}{k}, \quad |\beta_2| = r_2^{\frac{1}{2\varepsilon}} \left(\frac{p_2 b + q_2}{b-c}\right)^{\frac{\varepsilon+1}{\varepsilon}} (2An_2)^{\frac{\varepsilon+1}{\varepsilon}}.$$

Function $x_2(t)$ has exactly (n_2-1) zeros in (c,b). The expression for derivative is

$$x_2'(t) = \operatorname{sign}(\beta_2)|\beta_2|^{\frac{1}{\varepsilon+1}} r_2^{\frac{1}{2\varepsilon+2}}\, p_2 S\left(|\beta_2|^{\frac{\varepsilon}{\varepsilon+1}} r_2^{-\frac{1}{2\varepsilon+2}} \frac{t-c}{p_2 t + q_2}\right) + \\ + \beta_2 S'\left(|\beta_2|^{\frac{\varepsilon}{\varepsilon+1}} r_2^{-\frac{1}{2\varepsilon+2}} \frac{t-c}{p_2 t + q_2}\right) \frac{p_2 c + q_2}{p_2 t + q_2}.$$

One obtains that

$$x_1'(c) = \operatorname{sign}(\beta_1)|\beta_1|^{\frac{1}{\varepsilon+1}} r_1^{\frac{1}{2\varepsilon+2}}\, p_1 S\left(|\beta_1|^{\frac{\varepsilon}{\varepsilon+1}} r_1^{-\frac{1}{2\varepsilon+2}} \frac{c-a}{p_1 c + q_1}\right) + \\ + \beta_1 S'\left(|\beta_1|^{\frac{\varepsilon}{\varepsilon+1}} r_1^{-\frac{1}{2\varepsilon+2}} \frac{c-a}{p_1 c + q_1}\right) \frac{p_1 a + q_1}{p_1 c + q_1} = \\ = \operatorname{sign}(\beta_1)|\beta_1|^{\frac{1}{\varepsilon+1}} r_1^{\frac{1}{2\varepsilon+2}}\, p_1 S(2An_1) + \beta_1 S'(2An_1) \frac{p_1 a + q_1}{p_1 c + q_1} = \\ = \left| \begin{array}{c} \text{Notice that} \\ S(2An_1) = 0, \\ S'(2An_1) = (-1)^{n_1} \end{array} \right| = \beta_1 (-1)^{n_1} \frac{p_1 a + q_1}{p_1 c + q_1},$$

that is,

$$x_1'(c) = \beta_1 (-1)^{n_1} \frac{p_1 a + q_1}{p_1 c + q_1}.$$

Similarly $x_2'(c) = \beta_2$.

We can formulate now the result.

Proposition 2.3 *The relation $x_1'(c) = x_2'(c)$ holds iff*

$$\beta_2 = \beta_1(-1)^{n_1}\frac{p_1 a + q_1}{p_1 c + q_1}, \quad \textit{that is,} \quad \beta_2 = \beta_1(-1)^{n_1}\frac{\eta_0}{\eta_1} \tag{19}$$

or

$$\begin{cases} |\beta_2| = |\beta_1|\dfrac{p_1 a + q_1}{p_1 c + q_1}, \quad \textit{that is,} \quad |\beta_2| = |\beta_1|\dfrac{\eta_0}{\eta_1}, \\ \\ \operatorname{sign}(\beta_2) = (-1)^{n_1}\operatorname{sign}(\beta_1). \end{cases} \tag{20}$$

Let us discuss the first condition (20). One has that

$$|\beta_1|\frac{p_1 a + q_1}{p_1 \lambda + q_1} = |\beta_2|,$$

$$r_1^{\frac{1}{2\varepsilon}}\left(\frac{p_1 c + q_1}{c - a}\right)^{\frac{\varepsilon+1}{\varepsilon}}(2An_1)^{\frac{\varepsilon+1}{\varepsilon}}\frac{p_1 a + q_1}{p_1 c + q_1} = r_2^{\frac{1}{2\varepsilon}}\left(\frac{p_2 b + q_2}{b - c}\right)^{\frac{\varepsilon+1}{\varepsilon}}(2An_2)^{\frac{\varepsilon+1}{\varepsilon}},$$

$$\left(\frac{(p_1 a + q_1)^2(\varepsilon+1)}{k}\right)^{\frac{1}{2\varepsilon}}\left(\frac{p_1 c + q_1}{c - a}\right)^{\frac{\varepsilon+1}{\varepsilon}}(2An_1)^{\frac{\varepsilon+1}{\varepsilon}}\frac{p_1 a + q_1}{p_1 c + q_1} =$$

$$= \left(\frac{(p_2 c + q_2)^2(\varepsilon+1)}{k}\right)^{\frac{1}{2\varepsilon}}\left(\frac{p_2 b + q_2}{b - c}\right)^{\frac{\varepsilon+1}{\varepsilon}}(2An_2)^{\frac{\varepsilon+1}{\varepsilon}},$$

$$(p_1 a + q_1)^{\frac{1}{\varepsilon}}\left(\frac{p_1 c + q_1}{c - a}\right)^{\frac{\varepsilon+1}{\varepsilon}} n_1^{\frac{\varepsilon+1}{\varepsilon}}\frac{p_1 a + q_1}{p_1 c + q_1} = (p_2 c + q_2)^{\frac{1}{\varepsilon}}\left(\frac{p_2 b + q_2}{b - c}\right)^{\frac{\varepsilon+1}{\varepsilon}} n_2^{\frac{\varepsilon+1}{\varepsilon}}$$

$$(p_1 a + q_1)^{\frac{1}{\varepsilon}+1}\left(\frac{1}{c - a}\right)^{\frac{\varepsilon+1}{\varepsilon}} n_1^{\frac{\varepsilon+1}{\varepsilon}}(p_1 c + q_1)^{\frac{\varepsilon+1}{\varepsilon}-1} = (p_2 c + q_2)^{\frac{1}{\varepsilon}}\left(\frac{p_2 b + q_2}{b - c}\right)^{\frac{\varepsilon+1}{\varepsilon}} n_2^{\frac{\varepsilon+1}{\varepsilon}}$$

Recall that $p_1 c + q_1 = p_2 c + q_2$ and $\frac{\varepsilon+1}{\varepsilon} - 1 = \frac{1}{\varepsilon}$, then

$$(p_1 c + q_1)^{\frac{\varepsilon+1}{\varepsilon}-1} = (p_2 c + q_2)^{\frac{1}{\varepsilon}}.$$

Therefore

$$n_1^{\frac{\varepsilon+1}{\varepsilon}}\left(\frac{p_1 a + q_1}{c - a}\right)^{\frac{\varepsilon+1}{\varepsilon}} = n_2^{\frac{\varepsilon+1}{\varepsilon}}\left(\frac{p_2 b + q_2}{b - c}\right)^{\frac{\varepsilon+1}{\varepsilon}},$$

and

$$n_1\,\frac{p_1 a + q_1}{c - a} = n_2\,\frac{p_2 b + q_2}{b - c} \quad \text{that is,} \quad n_1\,\frac{\eta_0}{c - a} = n_2\,\frac{\eta_2}{b - c}. \tag{21}$$

If any of the conditions (21) holds then the condition (19) is satisfied.

We have proved the following statement.

Proposition 2.4 *A solution of the problem (15) can be composed of solutions $x_1(t)$ and $x_2(t)$ of the problems* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$) *respectively, iff*

$$\begin{cases} p_2 b + q_2 = \dfrac{n_1}{n_2}\,\dfrac{b-c}{c-a}\,(p_1 a + q_1), \\ \\ \operatorname{sign}(\beta_2) = (-1)^{n_1}\operatorname{sign}(\beta_1), \end{cases}$$

or, equivalently,

$$\begin{cases} \eta_2 = \dfrac{n_1}{n_2}\,\dfrac{b-c}{c-a}\,\eta_0, \\ \\ \operatorname{sign}(\beta_2) = (-1)^{n_1}\operatorname{sign}(\beta_1). \end{cases}$$

3 Principal and non-principal solutions

A solution of a linear second-order equation

$$x'' = -p(t)x \tag{22}$$

is called *non-oscillatory*, if it has at least finite number of zeros. Then, by Sturm zero separation theorem, any other solution also has at least finite number of zeros. Equation (22) is called therefore by *non-oscillatory* if all solutions (or, equivalently, one solution) are non-oscillatory. It is known ([7, Ch. XI, § 6]), that a non-oscillatory equation always possesses two solutions $x_0(t)$ and $x_1(t)$ with the properties

$$\int^{\infty} \frac{d\,t}{x_0^2(t)} = \infty, \tag{23}$$

and

$$\int^{\infty} \frac{d\,t}{x_1^2(t)} < \infty. \tag{24}$$

These solutions are called *principal* and *nonprincipal* ones respectively. We will show that the nonlinear equation (3) with increasing function $pt + a$ have solutions which satisfy either the condition (23) or (24). Thus they are nonlinear analogues of principal and non-principal solutions.

Consider the Cauchy problem (3), where $pt+q$ increases. Notice that solutions $x(t;\beta)$ for any nonzero β have only finite number of zeros t_i, which are given by the relations (10).

One has two cases to be analyzed.

Case 1. Suppose that the number $\frac{\rho}{p} = \lim\limits_{t\to+\infty} \rho\frac{t-a}{pt+q}$ is not a zero of the function $S(t)$. Then $x(t;\beta)$ has the zeros t_i, which can be found from the equations $\rho\frac{t-a}{pt+q} = \tau_i$, where $\tau_i = 2Ai$ are the first zeros of $S(t)$, less then the limit $\frac{\rho}{p}$. The function $x(t;\beta)$ is then linear at infinity, namely,

$$x(t;\beta) = \beta^{\frac{1}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pt+q) S\left(\frac{\rho}{p}\right).$$

Case 2. If $\frac{\rho}{p} = 2Ai$ for some i, then

$$\lim_{t\to+\infty} x(t;\beta) = (-1)^{i+1}\frac{\beta(pa+q)}{p}$$

or

$$\lim_{t\to+\infty} x(t;\beta) = (-1)^{i+1}\big(2Ai(pa+q)\big)^{\frac{\varepsilon+1}{\varepsilon}}\left(\frac{p^2(\varepsilon+1)}{k}\right)^{\frac{1}{2\varepsilon}}.$$

This can be shown by application of the l'Hospital rule to the formula (5).

We summarize the results of this section in the following statement.

Proposition 3.1 *There exist positive numbers β_i, $i = 1, 2, \dots$ such that the solutions $x(t;\beta_i)$ have exactly $i-1$ zeros in $(a,+\infty)$ and $\lim_{t\to+\infty} x(t;\beta_i) = const.$*

Solutions $x(t;\beta)$ for remaining positive values of β tend to linear functions as $t \to +\infty$. For any positive integer i there exist solutions $x(t;\beta)$, which have exactly $i-1$ zeros in $(a,+\infty)$ and tend to linear functions.

Notice that those solutions $x(t;\beta)$, which tend to constants, satisfy the condition (23). Other nontrivial solutions satisfy the condition (24).

4 Nehari numbers for non-autonomous equations

4.1 Brief overview of the Nehari theory

Consider differential equations

$$x'' + xF(t,x^2) = 0 \tag{25}$$

and

$$x'' + P(t)x + xF(t,x^2) = 0, \tag{26}$$

where

(A1) $F(t,s) \in C((0,+\infty)\times[0,+\infty), R)$;

(A2) $F(t,s) > 0$ for $t > 0$, $s > 0$;

(A3) $t_2^{-\varepsilon}F(t_2,s) > t_1^{-\varepsilon}F(t_1,s)$ for $0 \le t_1 < t_2 < \infty$, fixed $s > 0$ and some $\varepsilon > 0$.

A special case of equation (25) is that of the Emden – Fowler type equation

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0, \quad q \in C(R,(0,+\infty)). \tag{27}$$

The general theorem was proved in [3, Theorem 3.2].

Theorem 4.1 *Let Γ_n denote the class of functions $x(t)$ with the following properties: $x(t)$ is continuous and piecewise differentiable in $[a,b]$; $x(a_\nu) = 0(\nu = 0, 1\dots, n, n \ge 1$, where the a_ν are numbers such that $a = a_0 < a_1 < \dots < a_{n-1} = b$; for $\nu = 1,\dots,n$, but $x(t) \not\equiv 0$ in any interval $[a_{\nu-1}, a_\nu]$, and*

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\,dt = \int_{a_{\nu-1}}^{a_\nu} x^2(t)F(t,x^2(t))\,dt, \tag{28}$$

*where F is subject to the conditions **(A1)**– **(A3)**. Set $G(t,y) = \int_0^y F(t,s)\,ds$.*

The extremal problem

$$\int_a^b [x'^2 - G(t,x^2)]\,dt = \min = \lambda_n, \quad x(t) \in \Gamma_n \tag{29}$$

has a solution $x_n(t)$ whose derivative is continuous throughout $[a,b]$, and the characteristic values λ_n are strictly increasing with n. The function $x_n(t)$ has precisely $n-1$ zeros in (a,b), and it is a solution of the differential system

$$x'' + xF(t,x^2) = 0, \quad x(a) = x(b) = 0. \tag{30}$$

Proposition 4.1 ([3], Lemma 3.1) *The following properties of characteristic numbers $\lambda_n(a,b)$ are valid.*

1) If $a \le a_1 < b_1 \le b$, then $\lambda_n(a,b) \le \lambda_n(a_1,b_1)$;

2) and $\lambda_n(a_1,b_1) \to +\infty$ as $b_1 - a_1 \to 0$;

3) $\lambda_n(a,b)$ continuously depends on both a and b.

Proposition 4.2 ([3], [6]) *If $F(t,s) \le F_1(t,s)$ for $t \ge 0,\ s > 0$, then $\lambda_n(a,b) \ge \lambda'_n(a,b)$, where $\lambda'_n(a,b)$ is a characteristic value for the equation $x'' + xF_1(t,x^2) = 0$.*

4.2 Emden – Fowler equation

The Nehari theory is applicable to the Emden – Fowler type equations of the form (27).

Proposition (4.2) states as

Corollary 4.1 *If $q(t) \le q_1(t)$ for $t \ge 0$, then $\lambda_n(a,b) \ge \lambda'_n(a,b)$, where $\lambda_n(a,b)$ and $\lambda'_n(a,b)$ are characteristic values for equations (27) and $x'' + q_1(t)|x|^{2\varepsilon}x = 0$ respectively.*

The extremal problem (29) for the case of equation (27) takes the form:

$$H(x) = \int_a^b \left[x'^2 - (1+\varepsilon)^{-1} q(t) x^{2+2\varepsilon}\right] dt \to \inf \tag{31}$$

over all functions $x(t)$, which are continuous and piece-wise continuously differentiable in $[a,b]$; there exist numbers a_ν such that

$$a = a_0 < a_1 < \ldots < a_{n-1} = b;$$

for $\nu = 1, \ldots, n$, $x(a_\nu) = 0$ but $x \not\equiv 0$ in any $[a_{\nu-1}, a_\nu]$, and

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\,dt = \int_{a_{\nu-1}}^{a_\nu} q(t) x^2 |x|^{2\varepsilon}\,dt. \tag{32}$$

The respective extremal functions $x_n(t)$ are solutions of equation (27), vanish at the points $t = a$ and $t = b$, have exactly $n-1$ zeros in (a,b) and satisfy the condition

$$\int_a^b x'^2\,dt = \int_a^b q(t) x^2 |x|^{2\varepsilon}\,dt. \tag{33}$$

By combining (32) with (33) one gets

$$\lambda_n(a,b) = \min_{x\in\Gamma_n} H(x) = H(x_n) = \frac{\varepsilon}{1+\varepsilon}\int_a^b q(t)x^{2+2\varepsilon}\,dt = \frac{\varepsilon}{1+\varepsilon}\int_a^b x'^2(t)\,dt. \tag{34}$$

Thus the characteristic number $\lambda_n(a,b)$ is up to a constant the minimal value of $\int_a^b x'^2(t)\,dt$ over solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has } n-1 \text{ zeros in } (a,b). \tag{35}$$

We will call the characteristic numbers λ_n by Nehari numbers in the sequel.
Remark 4.1. Nehari numbers $\lambda_n(a,b)$ are not uniquely defined by the interval (a,b). It was shown in [6] that there exist equations of the type (27), which have more than one Nehari solution for certain a and b.

4.3 Monotone coefficients

Proposition 4.3 *The Nehari numbers for the problem*

$$x'' = -\frac{k}{(pt+q)^{2\varepsilon+4}}|x|^{2\varepsilon}x, \quad x(a) = 0, \quad x(b) = 0 \tag{36}$$

are given by

$$\lambda_n(a,b) = \frac{\varepsilon(\varepsilon+1)^{\frac{1}{\varepsilon}}}{\varepsilon+2}(2An)^{\frac{2\varepsilon+2}{\varepsilon}}k^{-\frac{1}{\varepsilon}}\left(\frac{(pa+q)(pb+q)}{b-a}\right)^{\frac{\varepsilon+2}{\varepsilon}}.$$

Proof. It was shown above that

$$\varepsilon(1+\varepsilon)^{-1}\int_a^b x'^2 dt = \lambda_n(a,b). \tag{37}$$

Notice also that

$$\int_{a_{i-1}}^{a_i} x'^2 dt = \int_{a_{i-1}}^{a_i} x^2F(t,x^2)dt = \int_{a_{i-1}}^{a_i} \frac{kx^{2\varepsilon+2}}{(pt+q)^{2\varepsilon+4}}\,dt. \tag{38}$$

Consider a solution $x(t)$ of the boundary value problem (36), which satisfies the relation

$$x(t) = \beta^{\frac{1}{\varepsilon+1}}r^{\frac{1}{2\varepsilon+2}}(pt+q)S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}}r^{-\frac{1}{2\varepsilon+2}}\frac{t-a}{pt+q}\right), \quad x'(0) = \beta \geq 0,$$

where r and A were defined earlier. This solution has exactly $(n-1)$ zero in $[a;b]$, if β satisfies (13).

We wish to compute the indefinite integral in the right side of (38). One has that

$$\int \frac{kx^{2\varepsilon+2}(t)}{(pt+q)^{2\varepsilon+4}}\,dt = k\int \frac{\left(\beta^{\frac{1}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}}(pt+q) S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right)\right)^{2\varepsilon+2} dt}{(pt+q)^{2\varepsilon+4}}$$

$$= k\int \frac{\beta^2 r (pt+q)^{2\varepsilon+2} S^{2\varepsilon+2}\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right) dt}{(pt+q)^{2\varepsilon+4}}$$

$$= k\beta^2 r \int \frac{S^{2\varepsilon+2}\left(\beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{t-a}{pt+q}\right) dt}{(pt+q)^2}$$

$$= \left| \begin{array}{c} \beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \dfrac{t-a}{pt+q} = z \\[2ex] \beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}}\, d\left(\dfrac{t-a}{pt+q}\right) = dz \\[2ex] d\left(\dfrac{t-a}{pt+q}\right) = \dfrac{(pa+q)dt}{(pt+q)^2} \\[2ex] \beta^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \dfrac{(pa+q)dt}{(pt+q)^2} = dz \\[2ex] dt = \beta^{-\frac{\varepsilon}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pa+q)^{-1}(pt+q)^2 dz \end{array} \right|$$

$$= k\beta^2 r \int \frac{S^{2\varepsilon+2}(z) \beta^{-\frac{\varepsilon}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pa+q)^{-1}(pt+q)^2 dz}{(pt+q)^2}$$

$$= k\beta^2 r \beta^{-\frac{\varepsilon}{\varepsilon+1}} r^{\frac{1}{2\varepsilon+2}} (pa+q)^{-1} \int S^{2\varepsilon+2}(z)dz.$$

Let us proceed with computation of the integral $\int S^{2\varepsilon+2}(z)dz$. Recall that

$$S'^2(z) = 1 - S^{2\varepsilon+2}(z).$$

Then

$$\int S'^2(z)dz = \int S'(z)S'(z)dz = \int S'(z)d\big(S(z)\big) = \int u dv$$

$$= \left| \begin{array}{ll} u = S'(z) & du = S''(z)dz \\ dv = dS & v = S \end{array} \right| = uv - \int v du$$

$$= S'(z)S(z) - \int S(z)S''(z)dz$$

$$= S'(z)S(z) - \int S(z)\big(-(\varepsilon+1)S^{2\varepsilon+1}(z)\big)dz$$

$$= S'(z)S(t) + (\varepsilon+1)\int S^{2\varepsilon+2}(z)dz$$

and

$$\int \bigl(1 - S^{2\varepsilon+2}(z)\bigr)dz = z - \int S^{2\varepsilon+2}(z)dz.$$

Therefore

$$S'(z)S(z) + (\varepsilon+1)\int S^{2\varepsilon+2}(z)dz = z - \int S^{2\varepsilon+2}(z)dz,$$

$$S'(z)S(z) - z = -(\varepsilon+2)\int S^{2\varepsilon+2}(z)dz,$$

and, finally,

$$\int S^{2\varepsilon+2}(z)dz = \frac{1}{\varepsilon+2}\bigl(z - S'(z)S(z)\bigr).$$

Thus

$$\begin{aligned}
\int \frac{kx^{2\varepsilon+2}(t)}{(pt+q)^{2\varepsilon+4}}\,dt &= k\beta^2 r\beta^{-\frac{\varepsilon}{\varepsilon+1}}r^{\frac{1}{2\varepsilon+2}}(pa+q)^{-1}\int S^{2\varepsilon+2}(z)dz \\
&= \left|\begin{array}{c}
k\beta^2 r\beta^{-\frac{\varepsilon}{\varepsilon+1}}r^{\frac{1}{2\varepsilon+2}}(pa+q)^{-1}\frac{1}{\varepsilon+2} = \\
= \frac{k\beta^{2-\frac{\varepsilon}{\varepsilon+1}}r^{1+\frac{1}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} \\
= \dfrac{k\beta^{\frac{\varepsilon+2}{\varepsilon+1}}r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)}
\end{array}\right| \\
&= \frac{k\beta^{\frac{\varepsilon+2}{\varepsilon+1}}r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)}\bigl(z - S'(z)S(z)\bigr) \\
&= \frac{k\beta^{\frac{\varepsilon+2}{\varepsilon+1}}r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)}\Biggl[\beta^{\frac{\varepsilon}{\varepsilon+1}}r^{-\frac{1}{2\varepsilon+2}}\frac{t-a}{pt+q} - \\
&\qquad -S'\left(\beta^{\frac{\varepsilon}{\varepsilon+1}}r^{-\frac{1}{2\varepsilon+2}}\frac{t-a}{pt+q}\right)\cdot S\left(\beta^{\frac{\varepsilon}{\varepsilon+1}}r^{-\frac{1}{2\varepsilon+2}}\frac{t-a}{pt+q}\right)\Biggr] = \Phi(t).
\end{aligned}$$

Then $\displaystyle\int_a^b \frac{kx^{2\varepsilon+2}(t)}{(pt+q)^{2\varepsilon+4}}\,dt = \Phi(b) - \Phi(a)$. We have got the formula

$$\lambda_n(a,b) = \frac{\varepsilon}{\varepsilon+1}\left[\Phi(b) - \Phi(a)\right] = \frac{\varepsilon}{\varepsilon+1}\left[\Phi(b) - 0\right] = \frac{\varepsilon}{\varepsilon+1}\Phi(b).$$

Final computation gives

$$\Phi(b) = \frac{k|\beta|^{\frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} \left[|\beta|^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{b-a}{pb+q} - \right.$$
$$\left. - S'\left(|\beta|^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{b-a}{pb+q} \right) S\left(|\beta|^{\frac{\varepsilon}{\varepsilon+1}} r^{-\frac{1}{2\varepsilon+2}} \frac{b-a}{pb+q} \right) \right]$$
$$= \frac{k|\beta|^{\frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} \left[2An - S'(2An)\, S(2An) \right]$$
$$= \frac{k|\beta|^{\frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} \left[2An - S'(2An)\, 0 \right] = \frac{k|\beta|^{\frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} 2An$$
$$= \frac{k r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} 2An \left(r^{\frac{1}{2\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+1}{\varepsilon}} (2An)^{\frac{\varepsilon+1}{\varepsilon}} \right)^{\frac{\varepsilon+2}{\varepsilon+1}}$$
$$= \frac{k r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} 2An \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+1}{\varepsilon} \cdot \frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{\varepsilon+2}{2\varepsilon(\varepsilon+1)}} (2An)^{\frac{\varepsilon+1}{\varepsilon} \cdot \frac{\varepsilon+2}{\varepsilon+1}}$$
$$= \frac{k r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} (2An)^{1+\frac{\varepsilon+2}{\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}} r^{\frac{\varepsilon+2}{2\varepsilon(\varepsilon+1)}}$$
$$= \frac{k}{(\varepsilon+2)(pa+q)} r^{\frac{2\varepsilon+3}{2\varepsilon+2} + \frac{\varepsilon+2}{2\varepsilon(\varepsilon+1)}} (2An)^{\frac{2\varepsilon+2}{\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}}$$
$$= \left| \begin{array}{c} r^{\frac{2\varepsilon+3}{2\varepsilon+2}} = r\, r^{\frac{1}{2\varepsilon+2}} \\ \\ r^{\frac{2\varepsilon+3}{2\varepsilon+2} + \frac{\varepsilon+2}{2\varepsilon(\varepsilon+1)}} = r\, r^{\frac{1}{2\varepsilon+2} + \frac{\varepsilon+2}{2\varepsilon(\varepsilon+1)}} = r\, r^{\frac{1}{\varepsilon}} \end{array} \right|$$
$$= \frac{k}{(\varepsilon+2)(pa+q)} r\, r^{\frac{1}{\varepsilon}} (2An)^{\frac{2\varepsilon+2}{\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}}.$$

$$\lambda_n(a,b) = \frac{\varepsilon}{\varepsilon+1}\Phi(b) = \frac{\varepsilon k}{(\varepsilon+1)(\varepsilon+2)(pa+q)} (2An)^{\frac{2\varepsilon+2}{\varepsilon}} r\, r^{\frac{1}{\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}} =$$
$$= \left| \begin{array}{ll} kr\, r^{\frac{1}{\varepsilon}} (pa+q)^{-1} & = k(pa+q)^{-1} r^{\frac{\varepsilon+1}{\varepsilon}} = k(pa+q)^{-1} \left[\frac{(pa+q)^2(\varepsilon+1)}{k} \right]^{\frac{\varepsilon+1}{\varepsilon}} = \\ & = k(pa+q)^{-1} (pa+q)^{2\frac{\varepsilon+1}{\varepsilon}} (\varepsilon+1)^{\frac{\varepsilon+1}{\varepsilon}} k^{-\frac{\varepsilon+1}{\varepsilon}} = \\ & = k^{1-\frac{\varepsilon+1}{\varepsilon}} (pa+q)^{-1+\frac{2\varepsilon+2}{\varepsilon}} (\varepsilon+1)^{\frac{\varepsilon+1}{\varepsilon}} = \\ & = k^{-\frac{1}{\varepsilon}} (pa+q)^{\frac{\varepsilon+2}{\varepsilon}} (\varepsilon+1)^{\frac{\varepsilon+1}{\varepsilon}} \end{array} \right|$$
$$= \frac{\varepsilon}{(\varepsilon+1)(\varepsilon+2)} (2An)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} (pa+q)^{\frac{\varepsilon+2}{\varepsilon}} (\varepsilon+1)^{\frac{\varepsilon+1}{\varepsilon}} \left(\frac{pb+q}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}}$$

$$= \frac{\varepsilon}{\varepsilon+2}(2An)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} (\varepsilon+1)^{-1+\frac{\varepsilon+1}{\varepsilon}} \left(\frac{(pa+q)(pb+q)}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}}$$

$$= \frac{\varepsilon(\varepsilon+1)^{\frac{1}{\varepsilon}}}{\varepsilon+2}(2An)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} \left(\frac{(pa+q)(pb+q)}{b-a} \right)^{\frac{\varepsilon+2}{\varepsilon}} . \qquad \blacktriangleleft$$

4.4 Final remarks

Remark 4.2. Denote by $X(t)$ a solution of the problem (14), smoothly composed of solutions $x_1(t)$ and $x_2(t)$ of the problems (17) and (18) respectively. Recall that x_1 and x_2 are supposed to have $n_1 - 1$ and $n_2 - 1$ zeros in the intervals (a, c) and (c, b) respectively. Since they are the only solutions with the prescribed number of zeros, certainly they are the Nehari solutions. The respective Nehari numbers are

$$\lambda_{n_1}(a,c) = \frac{\varepsilon(\varepsilon+1)^{\frac{1}{\varepsilon}}}{\varepsilon+2}(2An_1)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} \left(\frac{(p_1a+q_1)(p_1c+q_1)}{c-a} \right)^{\frac{\varepsilon+2}{\varepsilon}}$$

and

$$\lambda_{n_2}(c,b) = \frac{\varepsilon(\varepsilon+1)^{\frac{1}{\varepsilon}}}{\varepsilon+2}(2An_2)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} \left(\frac{(p_2c+q_2)(p_2b+q_2)}{b-c} \right)^{\frac{\varepsilon+2}{\varepsilon}} .$$

One can show by computing that the value $\Lambda_N := \varepsilon(1+\varepsilon)^{-1} \int_a^b X'^2(t)\, dt$ is given by

$$\Lambda_N = \lambda_{n_1}(a,c) + \lambda_{n_2}(c,b) = \frac{\varepsilon(\varepsilon+1)^{\frac{1}{\varepsilon}}}{\varepsilon+2}(2A)^{\frac{2\varepsilon+2}{\varepsilon}} k^{-\frac{1}{\varepsilon}} \left(\frac{(p_1a+q_1)(p_1\lambda+q_1)}{\lambda-a} \right)^{\frac{\varepsilon+2}{\varepsilon}} n_1^{\frac{\varepsilon+2}{\varepsilon}} (n_1+n_2),$$

where $N = n_1 + n_2$. Evidently

$$\Lambda_N \geq \lambda_N(a,b), \tag{39}$$

where $\lambda_N(a,b)$ is the respective Nehari number. It is very plausible that the equality is valid in (39).

Remark 4.3. It appears that for any $i = 1, 2, \dots, n$ the relations

$$\int_{t_{i-1}}^{t_i} x'^2 dt = \int_{t_{i-1}}^{t_i} x^2 F(t, x^2) dt = \Phi(t_i) - \Phi(t_{i-1}) = \frac{k|\beta|^{\frac{\varepsilon+2}{\varepsilon+1}} r^{\frac{2\varepsilon+3}{2\varepsilon+2}}}{(\varepsilon+2)(pa+q)} 2A = const,$$

hold, that is,

$$\int_{t_0}^{t_1} x'^2 dt = \int_{t_1}^{t_2} x'^2 dt = \dots = \int_{t_{n-1}}^{t_n} x'^2 dt,$$

where t_i are the zeros of a solution $x(t)$. In view of the monotonicity of the functions $t_i(\beta)$ a solution $x(t)$ with exactly $n - 1$ zero in the interval $[a, b]$ is unique. The reduction of $x(t)$ to any of the intervals $[t_{i-1}, t_i]$ is a Nehari solution also and it follows from (37) that $\lambda(t_{i-1}, t_i) = \varepsilon(1+\varepsilon)^{-1} \int_{t_{i-1}}^{t_i} x'^2(t)\, dt$. Thus a Nehari solution for the interval $[a, b]$, which has exactly $n - 1$ zeros in the interior and which relates to the Nehari number $\lambda_n(a, b)$, is combined of Nehari's solutions on the intervals $[t_{i-1}, t_i]$, which do not vanish in the open intervals (t_{i-1}, t_i) and which relate to the Nehari's numbers $\lambda(t_{i-1}, t_i)$. The relations

$$\lambda(a, t_1) = \lambda(t_1, t_2) = \ldots = \lambda(t_{n-1}, b) \tag{40}$$

hold. It is of interest either this is true for more general coefficients $q(t)$.

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А.С. Грицанс, Ф.Ж. Садырбаев. Явные решения неавтономных уравнений типа Эмдена – Фаулера.

Аннотация. Рассматриваются нелинейные уравнения вида

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad ' = \frac{d}{dt}, \quad \varepsilon > 0, \quad q \in C(R, (0, +\infty)),$$

где $q(t)$ – некоторые специальные t-зависимые функции, для которых явные формулы решений могут быть получены. Изучаются свойства решений, конструируются главные и неглавные (в смысле Хартмана) решения и вычисляются числа Нехари.

УДК 517.51 + 517.91

A. Gricāns, F. Sadirbajevs. Formulas Emdena – Faulera tipa nelineāro diferenciālvienādojumu atrisinājumiem.

Anotācija. Tiek apskatīti nelineārie diferenciālvienādojumi formā

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad ' = \frac{d}{dt}, \quad \varepsilon > 0, \quad q \in C(R, (0, +\infty)),$$

kur $q(t)$ – speciālas t-atkarīgas funkcijas, kurām eksistē atrisinājumu formulas. Tiek pētītas atrisinājumu īpašības, tiek konstruēti t. s. principiālie un neprincipiālie atrisinājumi. Izskaitļoti Nehari skaitļi, kuri dod atrisinājumu variācijas raksturojumu.

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