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Types of solutions of the second order Neumann problem: multiple solutions

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Summary. The Neumann boundary value problem $x'' = f(t, x, x')$ (i), $x'(0) = 0 = x'(1)$ (ii) is studied. Suppose that an equation $(L_2x)(t) := \frac{d}{dt}(p(t)\,x') + q(t)\,x = F(t, x, x')$ (iii), where F is bounded, is equivalent to (i) in some (t, x, x')-domain D and solutions of the quasi-linear problem (iii), (ii) satisfy the estimate $(t, x(t), x'(t)) \in D$ $\forall t \in [0, 1]$. We say then that the original problem (i), (ii) allows for L_2-quasilinearization in D. In this case it is solvable. We show that if the original problem allows for quasilinearization with respect to essentially different linear parts $(L_2x)(t)$, then it has multiple solutions. Illustrative examples are analyzed.

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1 Introduction

We consider the second order equation

$$x'' = f(t, x, x') \tag{1}$$

together with the Neumann boundary conditions

$$x'(0) = 0, \quad x'(1) = 0, \tag{2}$$

provided that f is continuous along with the partial derivatives f_x and $f_{x'}$.

Let us recall the general scheme employed usually to study the two-point second order boundary value problems. In the case of, say, the Dirichlet boundary conditions

$$x(0) = 0, \quad x(1) = 0 \tag{3}$$

the problem (1), (3) can be reduced under certain conditions to that for an equation

$$x'' = F(t, x, x'), \tag{4}$$

where F is bounded and coincides with f in some (t, x, x')-domain D. The solvability of (4), (3) follows from the well-known results ([1], [2]). If a solution $x(t)$ of the modified problem (4), (3) is such that $(t, x(t), x'(t)) \in D \; \forall t \in [0, 1]$, then $x(t)$ solves also the original problem (1), (3). Of course, the system (1), (3) must satisfy certain conditions, usually the upper and lower functions condition and the Nagumo type condition. The homogeneous linear problem (that is, $x'' = 0, \quad x(0) = x(1) = 0$ in the case of the Dirichlet problem) is required to have the trivial solution only.

In the case of the Neumann boundary conditions reduction to equation (4) is not appropriate since the homogeneous problem

$$x'' = 0, \quad x'(0) = x'(1) = 0$$

has a nontrivial solution. One may overcome this difficulty by trying to reduce the original problem to the quasi-linear one of the form

$$x'' + x = F(t, x, x'), \quad x'(0) = x'(1) = 0.$$

Notice that the homogeneous problem $x'' + x = 0, \quad x'(0) = x'(1) = 0$ has only the trivial solution.

Our intent in this paper is to study the Neumann boundary value problem by reducing equation (1) to the quasi-linear one of the form

$$\frac{d}{dt}(p(t)\, x') + q(t)\, x = F(t, x, x'), \tag{5}$$

where the right side F is bounded. We investigate the effect the linear part bears on solutions. It appears that if the original equation can be written in the form (5) for several essentially different linear left sides and certain estimates can be obtained for respective quasi-linear problems, then the problem (1), (2) has multiple solutions. We mean that the linear parts in representations (5) have different oscillatory types.

Let us mention several papers on the relevant subject, namely, [7], [8], [9], [11]. It was shown in [7], for example, that under certain conditions the BVP (1), (2) has a solution $\xi(t)$ such that the linear equation of variations with respect to $\xi(t)$ is disconjugate in $[0, 1]$. Recall that the linear second order equation is called disconjugate in some open interval, if the only solution with more than one zero in this interval, is the trivial one.

One may consult also [9], [12] – [14].

2 Quasi-linear boundary value problems

Consider the problem (5), (2), where $p\,(t) > 0$, $p, q \in C(I)$, $I := [0, 1]$, $F, F_x, F_{x'} \in C(I \times \mathbb{R}^2, \mathbb{R})$.

The result below is well known ((([1], [2]).

Theorem 2.1 *Suppose that the problem*

$$(L_2x)(t) := \frac{d}{dt}(p(t)x') + q(t)x = 0, \tag{6}$$

(2) has only the trivial solution and $F(t,x,x')$ in (5) is bounded.
Then the problem (5), (2) has a solution.

Lemma 2.1 *The Green's function for the problem (6), (2) is given by*

$$G(t,s) = \frac{1}{W}\begin{cases} \dfrac{u(t)v(s)}{p(s)}, & 0 \le t < s \le 1, \\ \\ \dfrac{u(s)v(t)}{p(s)}, & 0 \le s < t \le 1, \end{cases} \tag{7}$$

where $u(t)$ and $v(t)$ are linearly independent solutions of $(L_2x)(t) = 0$, which satisfy the initial conditions $x'(0) = 0$ and $x'(1) = 0$ respectively, $W(s) = u(s)v'(s) - v(s)u'(s)$.

Proof. Follows from [5, Ch. 3, § 27].

Lemma 2.2 *A set S of all solutions of the BVP (5), (2) is non-empty and compact in $C^1(I \times \mathbb{R}^2, \mathbb{R})$.*

Proof. The first assertion follows from Theorem 2.1.
Rewrite the problem (5), (2) in the integral form

$$x(t) = \int_0^1 G(t,s)\, F(s, x(s), x'(s))\, ds. \tag{8}$$

Respectively

$$x'(t) = \int_0^1 \frac{\partial}{\partial t} G(t,s)\, F(s, x(s), x'(s))\, ds. \tag{9}$$

If $|F(t,x,x')| \le M,\ \forall (t,x,x') \in I \times \mathbb{R}^2$, then

$$|x(t)| \le \Gamma \cdot M, \quad |x'(t)| \le \Gamma_1 \cdot M, \quad \forall t \in I,$$

where Γ and Γ_1 are bounds for $|G(t,s)|$ and $|\frac{\partial}{\partial t}G(t,s)|$ respectively. Then the set S is bounded in C^1-norm.

Let us show that the set S satisfies the equicontinuity condition. First consider the difference $x(t_2) - x(t_1)$. One has, by virtue of (7), that

$$W \cdot x(t_2) = v(t_2) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds + u(t_2) \int_{t_2}^1 \frac{v(s)}{p(s)} F(s)\, ds \tag{10}$$

and

$$W \cdot x(t_1) = v(t_1) \int_0^{t_1} \frac{u(s)}{p(s)} F(s)\, ds + u(t_1) \int_{t_1}^1 \frac{v(s)}{p(s)} F(s)\, ds, \tag{11}$$

where $F(s)$ stands for $F(s, x(s), x'(s))$.

It follows from (10) and (11) that

$$
\begin{aligned}
|W| \cdot |x(t_2) - x(t_1)| \;\; &\le \left| v(t_2) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds - v(t_1) \int_0^{t_1} \frac{u(s)}{p(s)} F(s)\, ds \right| \\
&+ \left| u(t_2) \int_{t_2}^{1} \frac{v(s)}{p(s)} F(s)\, ds - u(t_1) \int_{t_1}^{1} \frac{v(s)}{p(s)} F(s)\, ds \right| .
\end{aligned}
$$

Let us estimate each addend. It follows for the first one that

$$
\begin{aligned}
&\left| v(t_2) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds - v(t_1) \int_0^{t_1} \frac{u(s)}{p(s)} F(s)\, ds \right| \\
&= \Bigg| v(t_2) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds - v(t_1) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds \\
&+ v(t_1) \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds - v(t_1) \int_0^{t_1} \frac{u(s)}{p(s)} F(s)\, ds \Bigg| \\
&\le |v(t_2) - v(t_1)| \left| \int_0^{t_2} \frac{u(s)}{p(s)} F(s)\, ds \right| + |v(t_1)| \left| \int_{t_1}^{t_2} \frac{u(s)}{p(s)} F(s)\, ds \right| \\
&\le \frac{M}{p_0} \int_0^1 |u(s)|\, ds \cdot v'^* \cdot |t_2 - t_1| + v^* \cdot \frac{M}{p_0} \cdot u^* \cdot |t_2 - t_1| \\
&\le \frac{M}{p_0} \cdot u^*(v'^* + v^*) \cdot |t_2 - t_1|,
\end{aligned}
$$

where $p_0 = \min p(t), \quad u^* = \max |u(t)|, \quad v^* = \max |v(t)|, \quad v'^* = \max |v'(t)|, \;\; t \in I.$

The second addend can be estimated similarly.

One can show in analogous way that the set of functions $x'(t)$ is also equicontinuous, if $x \in S$.

Hence the proof. ■

Lemma 2.3 *There exists an element $x^* \in S$ with the property that $x^*(0) = \max\{x(0) : x \in S\}$. Similarly there exists an element $x_* \in S$ with the property that $x_*(0) = \min\{x(0) : x \in S\}$.*

Proof. First let us prove that the set $S_0 := \{x(0) : \; x \in S\}$ is compact in $\mathbb{R}$. We will show that the set above is bounded and closed. Boundedness follows from Lemma 2.2. Let us show that this set is closed. Suppose that $x_n(0) \to r$, where $x_n \in S$. Then, by compactness of the set S, one may find a subsequence x_{n_k} which tends to some $x \in S$ as $n_k \to +\infty$. Obviously $x(0) = r$. Thus $r \in S_0$.

Since one-dimensional closed sets have the minimal and the maximal elements, the proof follows. ■

Lemma 2.4 *All solutions of (5) are extendable to the interval* $[0,1]$ *and uniquely defined by the initial data.*

Proof. The first assertion follows from boundedness of F. Notice that since the continuous partial derivatives F_x and $F_{x'}$ exist, equation (5) satisfies the Lipschitz condition in any compact in $I \times \mathbb{R}^2$ domain. Then solutions of (5) are uniquely defined by the initial data and continuously depend on the initial data. ■

Consider the Cauchy problem

$$(L_2 x)(t) = 0, \quad x(0) = 1, \quad x'(0) = 0. \tag{12}$$

Definition 2.1 *We say that a linear part* $(L_2 x)(t)$ *is i-nonresonant in the interval* I *(with respect to the Neumann boundary conditions (2)), if a solution* $x(t)$ *of the problem (12) satisfies the conditions:*
a) zeros of $x(t)$ *and zeros of* $x'(t)$ *alternate in the interval* $[0,1)$;
b) there exist exactly i *points* $t_j \in (0,1)$ *such that* $x'(t_j) = 0$ *and* $x'(1) \neq 0$.

Denote by $x(t;\alpha)$ a solution of the Cauchy problem (1) (or (5)),

$$x(0) = \alpha, \quad x'(0) = 0. \tag{13}$$

Consider equation (5) with the initial conditions (13).

Lemma 2.5 *Suppose that* $(L_2 x)(t)$ *is i-nonresonant in the interval* I*. Let* $\xi(t)$ *be any element of* S*.*

Then for $\alpha \to \pm\infty$ *the difference* $u(t;\alpha) = x(t;\alpha) - \xi(t)$ *has exactly* i *points* $t_j \in (0,1)$ *such that* $u'(t_j;\alpha) = 0$ *and* $u'(1;\alpha) \neq 0$.

Proof. Consider the difference $u(t;\alpha) = x(t;\alpha) - \xi(t)$. One has that

$$(L_2 u)(t) = F(t,x,x') - F(t,\xi,\xi'), \quad u'(0) = 0,\ u(0) = \alpha - \xi(0).$$

Introduce new variable v by $v := \dfrac{u}{\alpha - \xi(0)}$. Then $v(t;\alpha)$ satisfies

$$(L_2 v)(t) = \frac{F(t,x,x') - F(t,\xi,\xi')}{\alpha - \xi(0)}, \tag{14}$$

$$v(0) = 1, \quad v'(0) = 0.$$

Since F is bounded the right side in (14) tends to zero as $\alpha \to \pm\infty$ uniformly in $t \in I$. Then by continuous dependence on the right side $v(t;\alpha)$ tends to a solution of the initial value problem (12). ■

Lemma 2.6 *Let* $\xi(t)$ *be any element of* S*.*

Zeros of the difference $u(t;\alpha) = x(t;\alpha) - \xi(t)$, $\alpha \neq \xi(0)$, *continuously depend on* α *in intervals of existence.*

Proof. Follows from continuous dependence of solutions of (5) on initial data and from the fact that $u(t;\alpha)$ cannot have double zeros. Indeed, if this were the case, then $x(t;\alpha) = \xi(t)$ and $x'(t;\alpha) = \xi'(t)$ at some point $t \in I$. Then $x \equiv \xi$, by the uniqueness of solutions of the Cauchy problem for (5). ■

3 Main results

Consider the problem (1), (2).

Definition 3.1 *Let equations (1) and (5), where the linear part in (5) is i-nonresonant in the interval I, be equivalent in a domain*

$$\Omega = \{(t, x, x') : \quad 0 \le t \le 1, \quad |x| < N, \quad |x'| < N_1\}. \tag{15}$$

Suppose that any solution $x(t)$ of the quasi-linear problem (5), (2) satisfies the estimate

$$|x(t)| < N, \quad |x'(t)| < N_1 \quad \forall t \in I. \tag{16}$$

We will say then that the problem (1), (2) allows for i-quasilinearization with respect to a domain Ω.

Remark 3.1. Examples of nonlinear problems which allow for quasilinearization will be given in the next section.

Theorem 3.1 *If the problem (1), (2) allows for i-quasilinearization with respect to some domain Ω, then it has a solution.*

Proof. Let $x(t)$ be a solution of the quasi-linear problem (5), (2). If $x(t)$ satisfies the estimate (16) and equations (1) and (5) are equivalent in Ω, then $x(t)$ solves also the problem (1), (2).

Theorem 3.2 *Suppose that the problem (1), (2) allows for i-quasilinearization with respect to a domain Ω_N and, at the same time, it allows for j-quasilinearization with respect to a domain Ω_M. It is assumed that $i \neq j$.*

Then the problem (1), (2) has at least 2 solutions.

Proof. Let $(L_2x)(t)$ and $(l_2x)(t)$ be respectively i-nonresonant and j-nonresonant linear parts. Equation (1) can be represented as

$$(L_2x)(t) = \Phi(t, x, x') \tag{17}$$

and

$$(l_2x)(t) = \Psi(t, x, x'). \tag{18}$$

Denote by $\xi(t)$ a solution x^* of the problem (17), (2). This solution exists by Lemma 2.3. Denote by $\eta(t)$ a solution x^* of the problem (18), (2). The solution $\xi(t)$ satisfies the estimate (16) and equation (17) is equivalent to equation (1) in the domain Ω_N. The solution $\eta(t)$ satisfies the estimate

$$|\eta(t)| < M, \quad |\eta'(t)| < M_1 \quad \forall t \in I. \tag{19}$$

and equation (18) is equivalent to equation (1) in the respective domain Ω_M.

Consider $\xi(t)$ and the difference $u(t; \alpha) = y(t; \alpha) - \xi(t)$, where $y(t; \alpha)$ is a solution of (17), which satisfies the initial conditions

$$y(0) = \xi(0) + \alpha, \quad y'(0) = \xi'(0) = 0. \tag{20}$$

By Lemma 2.5 the function $\frac{u(t;\alpha)}{\alpha-\xi(0)}$ tends to a solution of the problem (12) as $\alpha \to +\infty$. Then $u'(t;\alpha)$ for $\alpha \sim +\infty$ vanishes exactly at i points of the interval $(0,1)$ and $u'(1;\alpha) \neq 0$. Notice that the function of α $u'(1;\alpha)$ does not vanish for $\alpha \in (0,+\infty)$. If $u'(1;\alpha_1) = 0$ for some $\alpha_1 \in (0,+\infty)$ then the respective $y(t;\alpha_1)$ solves the problem (17), (2) and this contradicts the choice of $\xi(t)$ as a solution of (17), (2), which has the maximal derivative at the point $t = 0$. It follows from the arguments above that $u'(1;\alpha)$ is positive for $\alpha \sim 0$, if i is odd, and $u'(1;\alpha)$ is negative for $\alpha \sim 0$, if i is even.

Behavior of the difference $v(t;\alpha) = z(t;\alpha) - \eta(t)$, where $z(t;\alpha)$ is a solution of (18), which satisfies the initial conditions

$$z(0) = \eta(0) + \alpha, \quad z'(0) = \eta'(0) = 0 \tag{21}$$

can be analyzed in a similar way. The function $v'(1;\alpha)$ is also positive for $\alpha \sim 0$, if i is odd, and it is negative for $\alpha \sim 0$, if i is even.

Consider the case of i being even and j being odd. Suppose $\xi = \eta =: w$. By continuous dependendence of solutions of equations (17) and (18) on initial data, functions $x(t;\alpha)$ (and $y(t;\alpha) = x(t;\alpha)$) for α small enough satisfy both the estimates (16) and (19). Then $x(t;\alpha)$ are solutions of equation (1) too. One has then that $u(t;\alpha) \equiv v(t;\alpha)$ for small α. On the other hand, since i is even, $u'(1;\alpha) < 0$, and, since j is odd, $v'(1;\alpha) > 0$ for small α. The contradiction proves that ξ and η are different solutions of (1), (2).

Suppose now that both i and j are even different positive integers. To be definite, consider the case $i = 2$, $j = 4$. Let u and v have the same meaning as above. Consider the function $u(t;\alpha)$. The number of zeros of $u(t;\alpha)$ in the interval I for $\alpha \sim +\infty$ is either 2, or 3. The number of zeros is not greater than 3 for any $\alpha \in (0,+\infty)$. Indeed, if $u(t;\alpha)$ has 4-th zero in I for some $\alpha = \alpha_2$ then $u'(1;\alpha)$ vanishes for some $\alpha_3 > \alpha_2$. This is not possible since $x(t;\alpha_3)$ is then a solution of the problem (17), (2) and $x'(0;\alpha_3) > \xi'(0)$. Then the maximal number of zeros of the function $u(t;\alpha)$ is 3.

Consider the functions $v(t;\alpha)$. The number of zeros of $v(t;\alpha)$ in the interval I for $\alpha \sim +\infty$ is either 4, or 5. Let us show that the number of zeros of $v(t;\alpha)$ in the interval I is not less than 4 for any $\alpha \in (0,+\infty)$. Indeed, if $v(t;\alpha)$ has the 3-rd zero in I for some $\alpha = \alpha_4$ then $v'(1;\alpha)$ vanishes for some $\alpha_5 > \alpha_4$. This is not possible since $y(t;\alpha_5)$ is then a solution of the problem (18), (2) and $x'(0;\alpha_5) > \xi'(0)$. Then the minimal number of zeros of the function $v(t;\alpha)$ is 4.

Suppose that $\xi = \eta =: w$. By the arguments above $u(t;\alpha) \equiv v(t;\alpha)$ for small $\alpha > 0$. On the other hand, $u(t;\alpha)$ has at most three zeros in I and $v(t;\alpha)$ has at least four zeros in I for small α. The contradiction proves that ξ and η are different solutions of (1), (2).

Other cases can be treated similarly. ■

Corollary 3.1 *Suppose that the problem (1), (2) allows for i_j-quasilinearizations with respect to domains Ω_{N_j}, $j = 1, \ldots, n$, where $i_j \neq i_k$, if $j \neq k$.*

Then the problem (1), (2) has at least n solutions.

Proof. It follows from Theorem 3.1 that for any $j \in \{1, \ldots, n\}$ the problem (1), (2) has a solution x_j, associated with the respective i_j-nonresonant linear part. By Theorem 3.2 all such solutions are different. ■

4 Examples and applications

In this section we show that the quasilinearization scheme works for certain classes of equations.

Example 1. Consider the boundary value problem

$$x'' = -\lambda^2 \cdot |x|^p \operatorname{sign} \mathrm{x}, \qquad \mathrm{x}'(0) = \mathrm{x}'(1) = 0, \tag{22}$$

where $p > 0, \quad p \neq 1$.
The equivalent problem is

$$x'' + k^2 x = k^2 x - \lambda^2 \cdot |x|^p \operatorname{sign} \mathrm{x}, \qquad \mathrm{x}'(0) = \mathrm{x}'(1) = 0, \tag{23}$$

where k satisfies

$$i\pi < k < (i+1)\pi \tag{24}$$

for some i $(i = 0, 1, \ldots)$. This condition means that the linear part $x'' + k^2 x$ is i-nonresonant in I. The respective homogeneous problem

$$x'' + k^2 x = 0, \qquad x'(0) = x'(1) = 0, \tag{25}$$

has then only the trivial solution.

The Green's function of the problem (25) is given by

$$G_k(t,s) = \begin{cases} \dfrac{\cos k(s-1) \cdot \cos kt}{k \sin k}, & 0 \leq t \leq s \leq 1, \\[2ex] \dfrac{\cos k(t-1) \cdot \cos ks}{k \sin k}, & 0 \leq s \leq t \leq 1 \end{cases} \tag{26}$$

and satisfies the estimate

$$|G_k(t,s)| \leq \Gamma_k = \frac{1}{k \cdot |\sin k|}. \tag{27}$$

Consider the function $f_k(x) := k^2 x - \lambda^2 \cdot |x|^p \operatorname{sign} \mathrm{x}$. Since it is odd, we can treat it for positive values of x only. There exists a point of local extremum x_0. Calculation shows that

$$x_0 = \left(\frac{k^2}{\lambda^2 p}\right)^{\frac{1}{p-1}}.$$

In the case of $p > 1$ x_0 is a point of maximum and in the case of $0 < p < 1$ x_0 is the minimum point. Set

$$M_k = |f_k(t, x_0)| = \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \lambda^{\frac{2}{1-p}}. \tag{28}$$

The function $|f_k(x)|$ vanishes at $x_1 = (\frac{k^2}{\lambda^2})^{\frac{1}{p-1}}$ and unboundedly increases for $x > x_1$. There exists $x_2 \in (x_1, +\infty)$ such that $f_k(x_2) = -f_k(x_0)$. Set $N_k = x_2$. The number N_k can be represented as

$$N_k = \left(\frac{k^2}{\lambda^2}\right)^{\frac{1}{p-1}} \beta, \tag{29}$$

where β satisfies the equation

$$\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}. \tag{30}$$

The equation (30) has a root $\beta > 1$ for any positive p ($p \neq 1$).

One can consider the quasi-linear problem

$$x'' + k^2 x = F_k(x), \qquad x'(0) = x'(1) = 0, \tag{31}$$

instead of (23), where

$$F_k(x) := \varphi(x) \cdot f_k(x))$$

and $\varphi(x)$ is a bounded smooth function such that $\varphi(x) = 1$ for $|x| \leq N_k$, $\varphi(x) = 0$ for $|x| \geq N_k + \varepsilon$ and $\max\{|F_k| : x \in \mathbb{R}\} = M_k$. A number ε can be chosen arbitrarily small.

If the inequality

$$\Gamma_k \cdot M_k < N_k \tag{32}$$

holds, then the original problem has a solution similar to the linear part $x'' + k^2 x$.

If analogous quasilinearization is possible for k_j such that

$$i_j \pi < k_j < (i_j + 1)\pi, \qquad j = 1, ..., m$$

and the inequalities

$$\Gamma_{k_j} \cdot M_{k_j} < N_{k_j}, \tag{33}$$

hold, then the problem (31) has at least m solutions $x_1, ...x_m$. Any solution x_j is similar to the linear part $x'' + k_j^2 x$.

It follows from (27), (28), (29) that the inequality (32) takes the form

$$\frac{1}{k \cdot |\sin k|} \cdot \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \lambda^{\frac{2}{1-p}} \leq \left(\frac{k^2}{\lambda^2}\right)^{\frac{1}{p-1}} \beta,$$

$$\frac{k}{|\sin k|} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}. \tag{34}$$

Notice that the inequality (34) is independent of λ.
To simplify calculations, let $|\sin k| = 1$ (that is, $k = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$). Then the latter inequality reduces to

$$k < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}. \tag{35}$$

Proposition 4.1 *The linear part $(L_2 x)(t) := x'' + k^2 x$ is i-nonresonant in the interval I for any $k = \dfrac{\pi}{2} + \pi i$, $i = 0, 1, 2, \ldots$.*

Proof. By simple calculations.

Consider $\Omega = \{(t,x): \ 0 \le t \le 1, \quad |x| < N_k, \ |x'| < +\infty\}$. Let us examine for which values of k this problem allows for Ω-quasilinearization. Then the problem under consideration is solvable.

Computations show that for $p \in [0.5, 1) \cup (1, 2]$ there exist at least two values of k

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2},$$

which satisfy (35). Then at least two solutions of the problem (22) exist.

Since $\beta > 1$ and $\lim\limits_{p \to 1} \dfrac{p^{\frac{p}{p-1}}}{|p-1|} = +\infty$ the right side of the inequality (35) tends to ∞ as $p \to 1$. Then the inequality (35) holds for arbitrarily large values of $k = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$.

If

1) $p = \dfrac{1}{2}$ or $p = 2$ then the appropriate values of k (those which satisfy the inequality (35) are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2};$$

2) $p = \dfrac{2}{3}$ or $p = \dfrac{3}{2}$ then the appropriate values of k are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2}, \quad k_2 = \frac{5\pi}{2};$$

3) $p = \dfrac{3}{4}$ or $p = \dfrac{4}{3}$ then the appropriate values of k are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2}, \quad k_2 = \frac{5\pi}{2}, \quad k_3 = \frac{5\pi}{2}$$

and so on.

Proposition 4.2 *For any $p \in [0.5, 2]$, $\quad p \ne 1$, there exist at least two values of k of the form $k = \dfrac{\pi}{2} + \pi n$, $\quad n = 0, 1, 2, \ldots$, which satisfy the inequality* (35).

Therefore there exist at least two (different) solutions of the problem (22), *which satisfy the estimates $|x_k(t)| \le N_k$.*

Let us illustrate the proposition above by considering the specific case of $p = \dfrac{3}{2}$.

The problem

$$x'' = -|x|^{\frac{3}{2}} \operatorname{sign}(\mathrm{x}), \qquad \mathrm{x}'(0) = \mathrm{x}'(1) = 0 \tag{36}$$

can be reduced to

$$(x')^2 + \frac{4}{5}|x|^{\frac{5}{2}} = C,$$

where

$$C = const = \frac{4}{5}\big|x(0)\big|^{\frac{5}{2}} = \frac{4}{5}\big|x(1)\big|^{\frac{5}{2}} = \frac{4}{5}\big|x_{eks}\big|^{\frac{5}{2}}.$$

It follows from the equality above that solutions of the problem (22) possess the followinf properties:

$$|x(0)| = |x(1)| = |x_{eks}|,$$

where x_{eks} stand for an extremum of $x(t)$, $|x_{eks}| = \dfrac{25}{16}\dfrac{J^4}{\tau^4}$, where

$$J = \int_0^1 \frac{ds}{\sqrt{1 - s^{\frac{5}{2}}}} \approx 1.472,$$

τ is the length of the interval, where the function $x(t)$ varies from zero to an extremum. The respective quasilinear problem, as was mentioned above, is solvable for three different values of k. Respectively (29) $N_k = \beta\, k^4$, where $\beta \approx 1.250$. Denote by $\xi_k(t)$ solutions of the problem (36)

One has $k_0 = \dfrac{\pi}{2}$ $\xi_0 \equiv 0$.

For $k_1 = \dfrac{3\pi}{2}$ $\xi_1(t)$ the monotone in $[0, 1]$ function, which attends its extremum at the points $t = 0$, $t = 1$, the respective $\tau = \dfrac{1}{2}$. Then

$$\max_{[0,1]} |\xi_1(t)| \approx 117.374 < N_{\frac{3\pi}{2}} = \beta\Big(\frac{3\pi}{2}\Big)^4 \approx 616.415.$$

For $k_2 = \dfrac{5\pi}{2}$ the function $\xi_2(t)$ has exactly one extremum in $(0, 1)$, the respective $\tau = \dfrac{1}{4}$. Then

$$\max_{[0,1]} |\xi_2(t)| \approx 1877.981 < N_{\frac{5\pi}{2}} = \beta\Big(\frac{5\pi}{2}\Big)^4 \approx 4756.305.\blacksquare$$

Proposition 4.3 *For any positive integer m there exists $\varepsilon > 0$ such that if $p \neq 1$ satisfies the inequalities*

$$1 - \varepsilon < p < 1 + \varepsilon,$$

then $k = \dfrac{\pi}{2}(2n - 1)$, $\quad n = 1, 2, \ldots, m$ satisfy the inequality (35)*. Therefore there exist at least m (different) solutions of the problem* (22)*.*

Proof. Follows from (35).

Example 2. Consider the boundary value problem

$$x'' = -2mx' - \lambda^2 \cdot |x|^p \operatorname{sign}(\mathrm{x}), \tag{37}$$

$$x'(0) = x'(1) = 0, \tag{38}$$

where $m \neq 0, \quad p > 0, \quad p \neq 1.$

Equation (37) can be reduced to

$$\frac{d}{dt}\big(e^{2mt}\,x'\big) + e^{2mt}\,k^2 x = e^{2mt}\big(k^2 x - \lambda^2 \cdot |x|^p \,\mathrm{sign}\,(\mathrm{x})\big). \tag{39}$$

Denote $k^2 x - \alpha^2\,|x|^p \,\mathrm{sign}\,(\mathrm{x}) = \mathrm{f_k(x)}$, then

$$\frac{d}{dt}\big(e^{2mt}\,x'\big) + e^{2mt}\,k^2 x = e^{2mt}\,f_k(x). \tag{40}$$

Suppose that $k^2 > m^2$. Denote also $\sqrt{k^2 - m^2} =: r$.
The problem (40), (38) is equivalent to (37), (38). The respective homogeneous equation

$$\frac{d}{dt}\big(e^{2mt}\,x'\big) + e^{2mt}\,k^2 x = 0 \tag{41}$$

subject to prescribed boundary conditions (38) has only the trivial solution if the condition

$$W_0 := k^2 \sin r_{m,k} \neq 0 \tag{42}$$

holds. The Green's function $G_{m,k}(t,s)$ for the boundary value problem (41),(38) exists in the form

$$G_{m,k}(t,s) = \begin{cases} \dfrac{e^{-m(t+s)} \cdot v(s) \cdot u(t)}{W}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{e^{-m(t+s)} \cdot u(s) \cdot v(t)}{W}, & 0 \le s \le t \le 1, \end{cases} \tag{43}$$

where

$$W = r \cdot W_0,$$

and the functions $u(t)$ and $v(t)$ are such that

$$u_1(t) = e^{-mt} u(t)$$

is a solution of (41) and satisfies the condition

$$u_1'(0) = 0,$$

respectively

$$v_1(t) = e^{-m(t+1)} v(t)$$

is a solution of (41) and satisfies the condition

$$v_1'(1) = 0.$$

Functions $u(t)$ and $v(t)$ can be written in the form

$$u(t) = |k| \cdot \sin\big(rt + \varphi\big), \tag{44}$$

$$v(t) = |k| \cdot \sin\big(r(t-1) + \varphi\big), \tag{45}$$

where

$$\varphi = \arcsin \frac{r}{|k|}.$$

Evidently $u(t)$ and $v(t)$ can be estimated as

$$\big|u(t)\big| \leq |k|, \qquad \big|v(t)\big| \leq |k|. \tag{46}$$

The problem (40), (38) can be rewritten in integral form, that is,

$$x(t) = \int\limits_0^1 G_{m,k}(t,s) e^{2ms} f_k(x(s))\, ds, \tag{47}$$

Let us consider modified quasilinear equation

$$\frac{d}{dt}\big(e^{2mt}\, x'\big) + e^{2mt}\, k^2 x = e^{2mt}\, F_k(x), \tag{48}$$

where F_k is defined as in Example 1. The equivalent integral form is

$$x(t) = \int\limits_0^1 G_{m,k}(t,s) e^{2ms} F_k(x(s))\, ds. \tag{49}$$

If $F_k(x)$ satisfies the estimate

$$\max_{|x| \leq N_k} \Big|F_k(x)\Big| \leq M_k,$$

then

$$\big|x(t)\big| \leq \frac{k^2 e^{|m|} M_k}{|W|} = \frac{e^{|m|} M_k}{r \cdot |\sin r|}. \tag{50}$$

Proposition 4.4 *If for some m, k ($k^2 > m^2, \quad k^2 r \neq 0$) there exists N_k such that*

$$\forall x \quad \big|x(t)\big| < N_k \Rightarrow \max_{t \in [0,1]} \Big|F_k(x)\Big| \leq M_k$$

and the inequality

$$\frac{e^{|m|} M_k}{r \cdot |\sin r|} < N_k, \tag{51}$$

holds, then for m, k given a solution $x_{m,k}$ to the problem (40), (38) *exists which satisfies the estimate*

$$\big|x_{m,k}(t)\big| < N_k$$

Values of M_k and N_k can be found as in Example 1. We can set $r = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$. Then the inequality (51) takes the form

$$\frac{k^2 e^{|m|}}{r} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|},$$

$$\frac{(r^2+m^2)e^{|m|}}{r} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}, \tag{52}$$

where $r = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$.

In the table given in Appendix the results of calculations are provided. It is shown for certain values of p and m, which $r(r_0,\ r_1,\ \ldots)$ (and hence k) are good for the inequality (52) to be satisfied.

Proposition 4.5 *For given m there exists $\varepsilon > 0$ such that if $p \neq 1$ satisfies the inequalities*

$$1 - \varepsilon < p < 1 + \varepsilon,$$

then $r = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, \ldots q$ satisfy the inequality (52). Therefore there exist at least q (different) solutions of the problem (39), (38).

Proposition 4.6 *A linear part $(L_2x)(t) := \frac{d}{dt}\big(e^{2mt}\, x'\big) + e^{2mt}\, k^2 x$ for $m,\ k$ satisfy $\sqrt{k^2 - m^2} = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$ is n-nonresonant in the interval I.*

Proof. A solution of the problem

$$(L_2x)(t) = 0, \quad x(0) = 1, \quad x'(0) = 0, \quad x'(1) \neq 0$$

is given by

$$x(t) = e^{-mt}\big(\cos rt + \frac{m}{r}\sin rt\big),$$

where $r = \sqrt{k^2 - m^2}$. Since

$$x' = -e^{-mt}\frac{k^2}{r}\sin rt,$$

then $x'(t_j) = 0$, if

$$t_j = \frac{\pi\, l_j}{r},\ l_j \in \mathbb{N}.$$

Keeping in mind that r ($r = \dfrac{\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$), one gets that there exist exactly n points $t_j \in (0, 1)$ such that $x'(t_j) = 0$. ■

Proposition 4.7 *Let $\Omega = \{0 \le t \le 1, \quad |x| < N_k,\ |x'| < +\infty\}$, where $N_k = \left(\dfrac{k^2}{\lambda^2}\right)^{\frac{1}{p-1}}\beta$ and β satisfies the equation (30). Then if $m,\ k$ in (39) are such that the inequality*

$$\frac{k^2 e^{|m|}}{\sqrt{k^2 - m^2}} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|}$$

holds, the problem (37), (38) allows for k-quasilinearization with respect to Ω_{N_k} and therefore has a solution, which satisfies the estimate $|x| \le N_k$.

5 Appendix

	$\lvert m\rvert=\frac{3}{2}$	$\lvert m\rvert=1$	$\lvert m\rvert=\frac{1}{2}$	$\lvert m\rvert=\frac{1}{3}$
$p=\frac{2}{3}$	$-----$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$
$p=\frac{3}{4}$	$-----$	$r_0=\frac{\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$
$p=\frac{4}{5}$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$
$p=\frac{5}{6}$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$
$p=\frac{6}{7}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$
$p=\frac{7}{8}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_5=\frac{11\pi}{2}$
$p=\frac{8}{9}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_5=\frac{11\pi}{2}$	$r_0;\dots;r_6=\frac{13\pi}{2}$
$p=\frac{9}{10}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_6=\frac{13\pi}{2}$	$r_0;\dots;r_7=\frac{15\pi}{2}$
$p=\frac{10}{11}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_6=\frac{13\pi}{2}$	$r_0;\dots;r_8=\frac{17\pi}{2}$
$p=\frac{11}{12}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_7=\frac{15\pi}{2}$	$r_0;\dots;r_9=\frac{19\pi}{2}$
$p=\frac{12}{13}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_8=\frac{17\pi}{2}$	$r_0;\dots;r_9=\frac{19\pi}{2}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$p=\frac{13}{12}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_8=\frac{17\pi}{2}$	$r_0;\dots;r_9=\frac{19\pi}{2}$
$p=\frac{12}{11}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_7=\frac{15\pi}{2}$	$r_0;\dots;r_8=\frac{17\pi}{2}$
$p=\frac{11}{10}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_6=\frac{13\pi}{2}$	$r_0;\dots;r_7=\frac{15\pi}{2}$
$p=\frac{10}{9}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_5=\frac{11\pi}{2}$	$r_0;\dots;r_7=\frac{15\pi}{2}$
$p=\frac{9}{8}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_5=\frac{11\pi}{2}$	$r_0;\dots;r_6=\frac{13\pi}{2}$
$p=\frac{8}{7}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$	$r_0;\dots;r_5=\frac{11\pi}{2}$
$p=\frac{7}{6}$	$r_0=\frac{\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_4=\frac{9\pi}{2}$
$p=\frac{6}{5}$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$
$p=\frac{5}{4}$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$	$r_0;\dots;r_3=\frac{7\pi}{2}$
$p=\frac{4}{3}$	$-----$	$r_0=\frac{\pi}{2}$	$r_0;r_1=\frac{3\pi}{2}$	$r_0;r_1;r_2=\frac{5\pi}{2}$
$p=\frac{3}{2}$	$-----$	$r_0=\frac{\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$	$r_0=\frac{\pi}{2};r_1=\frac{3\pi}{2}$

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