Trigonometry of lemniscatic functions

A. Gritsans and F. Sadyrbaev

Summary. In this note we provide a set of useful formulae for the lemniscatic functions $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$. Various forms of the addition theorems are provided.

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1 Introduction

The lemniscatic functions can be characterized as "nonlinear" sin and cos functions. Recall that elementary trigonometric functions can be defined as linearly independent solutions of the ordinary differential equation

$$x'' = -x, \quad ' = \frac{d}{dt}. \tag{1}$$

Integration of (1) gives

$$x'^2 + x^2 = c^2, \tag{2}$$

where c is an arbitrary choice constant. It follows from (2), that $\sin t$ and $\cos t$ can be defined also as $t = \int\limits_0^{\sin t} \frac{ds}{\sqrt{1-s^2}}$ and $t = \int\limits_{\cos t}^{1} \frac{ds}{\sqrt{1-s^2}}$.

If one considers the nonlinear equation

$$x'' = -2x^3 \tag{3}$$

together with the initial conditions

$$x(0) = 0, \quad x'(0) = 1 \tag{4}$$

or

$$x'(0) = 0, \quad x(0) = 1, \tag{5}$$

then integration of (3) gives the relation

$$x'^2(t) + x^4(t) = 1. \tag{6}$$

Solutions $x(t)$ and $y(t)$ of the equation (3), which satisfy the initial conditions (4) and (5) respectively, can be defined as

$$t = \int_0^{x(t)} \frac{ds}{\sqrt{1-s^4}} \tag{7}$$

and

$$t = \int_{y(t)}^{1} \frac{ds}{\sqrt{1-s^4}} \tag{8}$$

for $t \in [0, A]$, where $A := \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. Functions defined by the integral relations (7) and (8) are known as *the lemniscatic functions* [10, § 22.8]. So $x(t)$ and $y(t)$ can be identified with $\operatorname{sl} t$ and $\operatorname{cl} t$ respectively (the notation $\operatorname{sl} t$ and $\operatorname{cl} t$ for the lemniscatic functions was introduced by C.F. Gauss).

One may consult the books [10], [4], [2], [8] for more information on the lemniscatic functions. Some recent applications of the lemniscatic functions in natural sciences can be found in [7].

2 Addition formulae

Likely as the elementary trigonometric functions, the lemniscatic functions allow for the addition formulae.

It follows from definition (7) that if we set

$$z = \int_0^r \frac{dt}{\sqrt{1-t^4}}, \quad x = \int_0^u \frac{dt}{\sqrt{1-t^4}}, \quad y = \int_0^v \frac{dt}{\sqrt{1-t^4}},$$

then

$$r = \operatorname{sl} z, \quad u = \operatorname{sl} x, \quad v = \operatorname{sl} y.$$

If one can find $r = r(u, v)$ such that the equality

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} = \int_0^u \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}}$$

holds, then $r = \operatorname{sl} z = \operatorname{sl}(x+y) = r(u, v) = r(\operatorname{sl} x, \operatorname{sl} y)$ and this relation can be interpreted as the addition formula for $\operatorname{sl} t$ at least in some neighborhood of $t = 0$. The right choice of r is ([4, Ch. 2, sec. 2.3])

$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}$$

and thus

$$\operatorname{sl}(x+y) = \frac{\operatorname{sl}(x)\sqrt{1-\operatorname{sl}^4(y)} + \operatorname{sl}(y)\sqrt{1-\operatorname{sl}^4(x)}}{1+\operatorname{sl}^2(x)\operatorname{sl}^2(y)}. \tag{9}$$

This is the first addition formula. It is true in some vicinity of zero. Later we modify it in such a way that it is valid for any x and y. Namely, the formulae

$$\mathrm{sl}(x+y) = \frac{\mathrm{sl}(x)\,\mathrm{sl}'(y) + \mathrm{sl}(y)\,\mathrm{sl}'(x)}{1+\mathrm{sl}^2(x)\,\mathrm{sl}^2(y)} \tag{10}$$

will be proved.

Although the lemniscatic functions were the first elliptic functions which were considered (investigations of the sum formula for $\mathrm{sl}\, t$ go back to Fagnano and L. Euler [4, §§ 2.1, 2.2, 2.3]), the easiest way to obtain (or reconstruct) the addition formulae for $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$ is to use those for the Jacobian elliptic functions $\mathrm{sn}(t;k)$, $\mathrm{cn}(t;k)$ and $\mathrm{dn}(t;k)$. It is known ([10, § 22.8]) that the lemniscatic functions can be expressed (at least in some neighborhood of $t=0$) also as [10, § 22.8]

$$\mathrm{sl}\, t = k\frac{\mathrm{sn}\,\frac{t}{k}}{\mathrm{dn}\,\frac{t}{k}}, \quad \mathrm{cl}\, t = \mathrm{cn}\,\frac{t}{k}, \quad k=\frac{1}{\sqrt{2}}. \tag{11}$$

It was shown in [3] that

$$\mathrm{sl}\,(\alpha+\beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{cl}(\beta) + \mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)}{1-\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)} \tag{12}$$

and

$$\mathrm{cl}(\alpha+\beta) = \frac{\mathrm{cl}(\beta)\,\mathrm{cl}(\alpha) - \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)}{1+\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}. \tag{13}$$

The proofs of (12) and (13) are based on the addition formulae for the Jacobian elliptic functions, which can be found, for example, in ([1], [5]).

Remark 2.1. Notice that the trigonometric analogues of (12) and (13) are $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Proposition 2.1 *The following relations are valid for any $t \in R$:*

$$\mathrm{sl}'(t) = \mathrm{cl}(t)\big(1+\mathrm{sl}^2(t)\big), \quad \mathrm{cl}'(t) = -\,\mathrm{sl}(t)\big(1+\mathrm{cl}^2(t)\big),$$

$$\mathrm{sl}'^2(t) + \mathrm{sl}^4(t) = 1, \quad \mathrm{cl}'^2(t) + \mathrm{cl}^4(t) = 1, \quad \mathrm{sl}^2(t) + \mathrm{sl}^2(t)\,\mathrm{cl}^2(t) + \mathrm{cl}^2(t) = 1,$$

$$\mathrm{sl}(t+A) = \mathrm{cl}(t), \quad \mathrm{cl}(t+A) = -\,\mathrm{sl}(t), \ \textit{where } A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}.$$

Proof. Proofs can be found in [3, Propositions 7.4, 7.5 and 7.6, Corollary 7.3].

Proposition 2.2

$$\mathrm{sl}\,(\alpha+\beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{sl}'(\beta) + \mathrm{sl}'(\alpha)\,\mathrm{sl}(\beta)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)} \tag{14}$$

Proof. Denote $\mathrm{sl}(\alpha) =: S_1$, $\mathrm{sl}(\beta) =: S_2$, $\mathrm{cl}(\alpha) =: C_1$, $\mathrm{cl}(\beta) =: C_2$. We mean also that $\mathrm{sl}'(\alpha) =: S_1'$, $\mathrm{sl}'(\beta) =: S_2'$, $\mathrm{cl}'(\alpha) =: C_1'$, $\mathrm{cl}'(\beta) =: C_2'$.

$$
\begin{aligned}
\mathrm{sl}(\alpha+\beta) &= \frac{S_1C_2+S_2C_1}{1-S_1S_2C_1C_2} = \frac{S_1C_2+S_2C_1}{1-S_1S_2C_1C_2}\cdot\frac{\big(1+S_1^2\big)\big(1+S_2^2\big)}{\big(1+S_1^2\big)\big(1+S_2^2\big)} \\
&= \frac{S_1C_2\big(1+S_1^2\big)\big(1+S_2^2\big)+S_2C_1\big(1+S_1^2\big)\big(1+S_2^2\big)}{\big(1+S_1^2\big)\big(1+S_2^2\big)-S_1S_2C_1C_2\big(1+S_1^2\big)\big(1+S_2^2\big)} \\
&= \frac{S_1\big(1+S_1^2\big)S_2'+S_2\big(1+S_2^2\big)S_1'}{\big(1+S_1^2\big)\big(1+S_2^2\big)-S_1S_2S_1'S_2'} \\
&= \frac{S_1\big(1+S_1^2\big)S_2'+S_2\big(1+S_2^2\big)S_1'}{\big(1+S_1^2\big)\big(1+S_2^2\big)-S_1S_2S_1'S_2'}\cdot\frac{\big(1+S_1^2\big)\big(1+S_2^2\big)+S_1S_2S_1'S_2'}{\big(1+S_1^2\big)\big(1+S_2^2\big)+S_1S_2S_1'S_2'}.
\end{aligned}
$$

Transformation of the numerator follows:

$$
\begin{aligned}
&\big[S_1\big(1+S_1^2\big)S_2'+S_2\big(1+S_2^2\big)S_1'\big]\cdot\big[\big(1+S_1^2\big)\big(1+S_2^2\big)+S_1S_2S_1'S_2'\big] \\
&= S_1\big(1+S_1^2\big)S_2'\big(1+S_1^2\big)\big(1+S_2^2\big)+S_2\big(1+S_2^2\big)S_1'\big(1+S_1^2\big)\big(1+S_2^2\big) \\
&+S_1\big(1+S_1^2\big)S_2'S_1S_2S_1'S_2'+S_2\big(1+S_2^2\big)S_1'S_1S_2S_1'S_2' \\
&\qquad\left|\begin{array}{c}\text{make use of}\\ S'^{\,2}=1-S^4=\big(1+S^2\big)\big(1-S^2\big)\end{array}\right| \\
&= S_1\big(1+S_1^2\big)S_2'\big(1+S_1^2\big)\big(1+S_2^2\big)+S_2\big(1+S_2^2\big)S_1'\big(1+S_1^2\big)\big(1+S_2^2\big) \\
&+S_1^2\big(1+S_1^2\big)\big(1-S_2^2\big)\big(1+S_2^2\big)S_2S_1'+S_2^2\big(1+S_2^2\big)\big(1-S_1^2\big)\big(1+S_1^2\big)S_1S_2' \\
&= S_2S_1'\big(1+S_1^2\big)\big(1+S_2^2\big)\left[1+S_2^2+S_1^2\big(1-S_2^2\big)\right]+S_1S_2'\big(1+S_1^2\big)\big(1+S_2^2\big) \\
&\times\left[1+S_1^2+S_2^2\big(1-S_1^2\big)\right]=S_2S_1'\big(1+S_1^2\big)\big(1+S_2^2\big)\left[1+S_2^2+S_1^2-S_1^2S_2^2\right] \\
&+S_1S_2'\big(1+S_1^2\big)\big(1+S_2^2\big)\cdot\left[1+S_1^2+S_2^2-S_2^2S_1^2\right] \\
&= \big(1+S_1^2\big)\big(1+S_2^2\big)\cdot\left[1+S_1^2+S_2^2-S_2^2S_1^2\right]\big(S_1S_2'+S_2S_1'\big).
\end{aligned}
$$

Transformation of the denominator follows:

$$\left(1+S_1^2\right)^2\left(1+S_2^2\right)^2-S_1^2S_2^2S_1'^2S_2'^2=$$

$$\left|\begin{array}{c}\text{make use of}\\ S'^2=1-S^4=\left(1+S^2\right)\left(1-S^2\right)\end{array}\right|$$

$$=\left(1+S_1^2\right)^2\left(1+S_2^2\right)^2-S_1^2S_2^2\left(1+S_1^2\right)\left(1-S_1^2\right)\left(1+S_2^2\right)\left(1-S_2^2\right)$$

$$=\left(1+S_1^2\right)\left(1+S_2^2\right)\left[\left(1+S_1^2\right)\left(1+S_2^2\right)-S_1^2S_2^2\left(1-S_1^2\right)\left(1-S_2^2\right)\right]$$

$$=\left(1+S_1^2\right)\left(1+S_2^2\right)\left[\left(1+S_1^2\right)\left(1+S_2^2\right)-S_1^2S_2^2\left(1-S_1^2\right)\left(1-S_2^2\right)\right]$$

$$=\left(1+S_1^2\right)\left(1+S_2^2\right)(1+S_1^2S_2^2)(1+S_1^2+S_2^2-S_1^2S_2^2).$$

Therefore

$$\operatorname{sl}(\alpha+\beta)=\frac{S_1S_2'+S_2S_1'}{1+S_1^2S_2^2}=\frac{\operatorname{sl}(\alpha)\operatorname{sl}'(\beta)+\operatorname{sl}'(\alpha)\operatorname{sl}(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)},$$

q.e.d.

Proposition 2.3

$$\operatorname{sl}(\alpha+\beta)=-\frac{\operatorname{cl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{cl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)} \tag{15}$$

Proof. One gets using the notation of the previous statement that

$$\operatorname{sl}(\alpha+\beta)=\frac{S_1S_2'+S_2S_1'}{1+S_1^2S_2^2}=\frac{S_1C_2(1+S_2^2)+S_2C_1(1+S_1^2)}{1+\frac{1-C_1^2}{1+C_1^2}\frac{1-C_2^2}{1+C_2^2}}$$

$$=\frac{S_1C_2(1+S_2^2)(1+C_1^2)(1+C_2^2)+S_2C_1(1+S_1^2)(1+C_1^2)(1+C_2^2)}{(1+C_1^2)(1+C_2^2)+(1-C_1^2)(1-C_2^2)}$$

$$\left|\begin{array}{c}\text{take into account that}\\ (1+S_i^2)(1+C_i^2)=2,\quad i=1,2\end{array}\right|$$

$$=\frac{2S_1C_2(1+C_1^2)+2S_2C_1(1+C_2^2)}{2+2C_1^2C_2^2}=\frac{S_1C_2(1+C_1^2)+S_2C_1(1+C_2^2)}{1+C_1^2C_2^2}$$

$$=-\frac{C_1'C_2+C_2'C_1}{1+C_1^2C_2^2}=-\frac{\operatorname{cl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{cl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)}.$$

Proposition 2.4

$$\operatorname{cl}(\alpha+\beta)=\frac{\operatorname{sl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{sl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)} \tag{16}$$

Proof.

$$\operatorname{cl}(\alpha+\beta) = \operatorname{sl}\big((\alpha+\beta)+A\big) = \operatorname{sl}\left((\beta+A)+\alpha\right) = \frac{\operatorname{sl}(\beta+A)\operatorname{sl}'(\alpha)+\operatorname{sl}(\alpha)\operatorname{sl}'(\beta+A)}{1+\operatorname{sl}^2(\beta+A)\operatorname{sl}^2(\alpha)}$$
$$= \left| \begin{array}{c} \operatorname{sl}(\beta+A)=\operatorname{cl}(\beta) \\ \operatorname{sl}'(\beta+A)=\operatorname{cl}'(\beta) \end{array} \right| = \frac{\operatorname{sl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{sl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)}.$$

3 Appendix

We summarize here the addition formulae for the lemniscatic functions together with their trigonometric analogues. The double argument formulae are provided also which can be obtained from the summation ones. The addition formulae are given in the "±"-form, which easily follow from the "+" formulae taking into account the evenness properties of $\operatorname{sl} t$ (odd function) and $\operatorname{cl} t$ (even function).

$$\operatorname{sl}(\alpha\pm\beta) = \frac{\operatorname{sl}(\alpha)\operatorname{cl}(\beta)\pm\operatorname{sl}(\beta)\operatorname{cl}(\alpha)}{1\mp\operatorname{sl}(\alpha)\operatorname{sl}(\beta)\operatorname{cl}(\alpha)\operatorname{cl}(\beta)}$$

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta\pm\sin\beta\cos\alpha$$

$$\operatorname{sl}(\alpha\pm\beta) = \frac{\operatorname{sl}(\alpha)\operatorname{sl}'(\beta)\pm\operatorname{sl}'(\alpha)\operatorname{sl}(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)}$$

$$\sin(\alpha\pm\beta) = \sin\alpha\sin'\beta\pm\sin'\alpha\sin\beta$$

$$\operatorname{sl}(\alpha\pm\beta) = \mp\frac{\operatorname{cl}(\alpha)\operatorname{cl}'(\beta)\pm\operatorname{cl}'(\alpha)\operatorname{cl}(\beta)}{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)}$$

$$\sin(\alpha+\beta) = \mp(\cos\alpha\cos'\beta\pm\cos'\alpha\cos\beta)$$

$$\operatorname{cl}(\alpha\pm\beta) = \frac{\operatorname{cl}(\alpha)\operatorname{cl}(\beta)\mp\operatorname{sl}(\alpha)\operatorname{sl}(\beta)}{1\pm\operatorname{sl}(\alpha)\operatorname{sl}(\beta)\operatorname{cl}(\alpha)\operatorname{cl}(u)}$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\operatorname{cl}(\alpha\pm\beta) = \frac{\operatorname{sl}'(\alpha)\operatorname{cl}(\beta)\pm\operatorname{sl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)}$$

$$\cos(\alpha\pm\beta) = \sin'\alpha\cos\beta\pm\sin\alpha\cos'\beta$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)}{1-\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)}{\operatorname{sl}^2(\alpha)+\operatorname{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{sl}'(\alpha)}{1+\mathrm{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\sin'\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)\big(1+\mathrm{sl}^2(\alpha)\big)}{1+\mathrm{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{-2\,\mathrm{cl}(\alpha)\,\mathrm{cl}'(\alpha)}{1+\mathrm{cl}^4(\alpha)}$$

$$\sin 2\alpha = -2\cos\alpha\cos'\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)\big(1+\mathrm{cl}^2(\alpha)\big)}{1+\mathrm{cl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{cl}(2\alpha) = \frac{\mathrm{cl}^2(\alpha)-\mathrm{sl}^2(\alpha)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\alpha)}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\mathrm{cl}(2\alpha) = \frac{\mathrm{cl}(\alpha)\,\mathrm{sl}'(\alpha)+\mathrm{sl}(\alpha)\,\mathrm{cl}'(\alpha)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\alpha)}$$

$$\cos 2\alpha = \cos\alpha\sin'\alpha + \sin\alpha\cos'\alpha$$

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А.С. Грицанс, Ф.Ж. Садырбаев. Тригонометрия лемнискатных функций.

Аннотация. Для лемнискатных функций $\mathrm{sl}\,t$ и $\mathrm{cl}\,t$ приводится набор формул, имеющих аналоги в теории элементарных тригонометрических функций. Даны различные варианты теорем сложения.

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A. Gricāns, F. Sadirbajevs. Lemniskātisko funkciju trigonometrija.

Anotācija. Tiek apskatīta virkne formulu, kas saista lemniskātiskās funkcijas $\mathrm{sl}\,t$ un $\mathrm{cl}\,t$ un kurām eksistē analogi elementāro trigonometrisko funkciju teorijā. Uzrādīti vairāki ekvivalenti saskaitīšanas teorēmas formulējumi.

Institute of Mathematics
and Computer Science,
University of Latvia
Riga, Rainis blvd 29

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Daugavpils University
Department of Natural Sciences
and Mathematics
Daugavpils, Parades str. 1