Differential Equations with Exponentially Growing Nonlinearities

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Summary. Differential equations of the type $x'' + g(x) = f(t, x, x')$ are considered, where $g(x)$ is an exponentially growing nonlinearity and f is a "slow growth" function. We derive conditions on g and f, which ensure the superlinear behaviour of solutions. The Dirichlet boundary value problem, for example, has infinitely many solutions.

MSC 34B15

1 Introduction

Consider equation

$$x'' + g(x) = 0, \quad '' := \frac{d^2}{dt^2}, \quad t \in [a, b], \tag{1}$$

where $g \in C^1(R)$, and the conditions

(A1) $x\, g(x) > 0$ for large in modulus values of x;

(A2) $\frac{g(x)}{x} \to +\infty$ as $|x| \to \infty$

are satisfied. Solutions of (1) with the initial values

$$x(a) = x_0, \quad x'(a) = y_0 \tag{2}$$

have zeros in the interval (a, b), if the number $\rho^2(a) = x_0^2 + y_0^2$ is large enough, and the number of zeros unboundedly increases together with $\rho^2(a)$. It can be shown that then the boundary value problem (1),

$$x(a) = 0, \quad x(b) = 0 \tag{3}$$

has infinitely many solutions.

Consider now perturbed equation

$$x'' + g(x) = f(t, x, x'), \tag{4}$$

where f is a continuous function together with the partial derivatives f_x and $f_{x'}$.

It is used to say that equation (1) is *superlinear* if the condition (A2) holds. Superlinear equations and associated boundary value problems were studied, for example, in the works [1] - [5]. The characteristic property of superlinear equations is formulated below.

Definition. We say that (4) possesses *the S-property* in a finite interval $[a,b]$ if (i) all solutions extend to $[a,b]$ and (ii) for any N there exists Δ such that the number of zeros in $[a,b]$ of any solution $x(t)$ of the initial value problem (4), (2) is greater than N, if $x_0^2+y_0^2>\Delta^2$.

It was shown in the work [1] that equation (4), where $g(x)$ meets tha condition (A2) is superlinear if the right side $f(t,x,x')$ is a linear function with respect to x and x'. Further generalizations may be found in the works by the authors [4] and [5].

It was shown in [5] that (4) is superlinear if $g(x)$ and $f(t,x,x')$ are as above and in addition to the condition (A1) the following conditions are satisfied:

(B1) $|g(x)|\geq k^2|x|^\gamma$ for large in modulus x, $\gamma>1$;

(B2) $|f(t,x,y)|\leq c\cdot y^{\frac{2\gamma}{\gamma+1}}$, for $t\in[a,b]$, x^2+y^2 large enough and the number $k^2>c_*:=\dfrac{c}{\gamma+1}\left[\dfrac{2\gamma c}{(\gamma+1)^2}\right]^\gamma$.

The estimates in (B1) and (B2) are the best possible.

In the sequel we consider the case of exponentially growing functions $g(x)$. One may expect then that the class of the right sides f is broader now in order the equation (4) to be superlinear.

2 Slow growing right sides

Theorem 1 *Equation*

$$x''+g(x)=c(x)\cdot x'^{2-\varepsilon}, \tag{5}$$

where $g(x)$ satisfies the condition (A1) and

(A3) $\dfrac{|g(x)|}{\exp(\kappa|x|)}\geq K$ *for x large in modulus, $K>0$ and $\kappa>0$;*

(A4) $c(x)\in C(R)$ *and $|c(x)|\leq c_*$ for any $x\in R$,*

possesses the S-property for any $2>\varepsilon>0$.

Proof: Define γ from the equality $2-\varepsilon=2\frac{\gamma}{\gamma+1}$. Evidently

$$K\ \exp(\kappa|x|)>k^2|x|^\gamma$$

for large in modulus x, where a constant k^2 is such that

$$k^2>\frac{c_*}{\gamma+1}\left[\frac{2\gamma c_*}{(\gamma+1)^2}\right]^\gamma.$$

Thus the conditions of Theorem 4.1 in [5] (listed in Introduction) are fulfilled and the result follows.

3 Fast growing right sides

Consider equation (4).

Theorem 2 *Suppose $g(x)$ satisfies the conditions (A1) and (A2). Assume also that one of the inequalities*

(A5) $f(t,x,y) \geq c(x) \cdot |y|^{2+\varepsilon}$;

(A6) $f(t,x,y) \leq -c(x) \cdot |y|^{2+\varepsilon}$

holds for large $x^2 + y^2$, where $c(x) \in C(R,(0,+\infty))$.

Then equation (4) does not possess the S-property for any $\varepsilon > 0$.

Proof: Consider the Cauchy problem

$$z'_x = p(x)z^{1+\delta} - q(x), \quad z(0) = z_0, \tag{6}$$

where $p(x) \in C(R,(0,+\infty))$, $q(x) \in C(R)$ and $\delta > 0$. If z_0 is sufficiently large, then $z(x)$ grows unboundedly as x increases.

Return to equation (4). Suppose the condition (A5) fulfills. Consider a solution $x(t)$ of (4) which satisfies the initial conditions

$$x(a) = 0, \quad x'(a) = \Delta. \tag{7}$$

We have then that

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -g(x(t)) + f(t,x(t),x'(t)), \\ x(a) = 0, \quad y(a) = \Delta, \end{cases}$$

or

$$\frac{dz}{dx} = 2\left[f(t,x(t),x'(t)) - g(x(t))\right], \tag{8}$$

where $z = x'^2$. One gets then using the inequality (A5) that

$$z'_x \geq 2c(x(t))z^{1+\frac{\varepsilon}{2}} - 2g(x(t)), \quad z(0) = \Delta^2. \tag{9}$$

Comparing (9) and (6) leads to a conclusion that $z(x(t))$ increases unboundedly as t increases if $\Delta > 0$ is large enough. This means that $x'(t)$ cannot vanish if $\Delta > 0$ is large. Thus equation (6) does not possess the S-property.

The case (A5) can be considered similarly. It reflects the situation when $x'(a) < 0$ is large enough in modulus.

4 Quadratic right sides

Consider equation

$$x'' + g(x) = c(x) \cdot x'^2, \tag{10}$$

where $g(x)$ satisfies the conditions (A1) and (A2). Suppose $c(x) \in C(R,(0,+\infty))$. If both $g(x)$ and $c(x)$ are exponentially growing functions then they may be in a conflict and equation (10) may not to be superlinear. We can prove the following result.

Theorem 3 *Suppose that $g(x)$ satisfies the conditions (A1) and (A2) and $c(x) \in C(R,(0,+\infty))$. If*

$$\int^{+\infty} e^{-2C(\xi)} g(\xi)\, d\xi < +\infty, \tag{11}$$

where $C(x) = \int_0^x c(s)\, ds$, then equation (10) does not possess the S-property.

Proof: Consider equation (10) with the initial conditions

$$x(a) = m > 0, \quad x'(a) = 0. \tag{12}$$

Let $x(t)$ be a solution of the Cauchy problem (10), (12). Denote by $T(m)$ the difference $t_1 - a$, where t_1 is a first zero of $x(t)$ in the interval (a, b). We will show that $T(m)$ is separated from zero if the condition (11) fulfils.

Consider the equivalent system

$$\begin{cases} x' = y, \\ y' = -g(x) + c(x) \cdot y^2, \\ x(a) = m, \quad y(a) = 0, \end{cases}$$

which reduces to

$$\frac{dz}{dx} = 2c(x)z - 2g(x), \quad z(m) = 0, \tag{13}$$

where $z = x'^2$. One obtains by resolving (13) that

$$z(x) = e^{2C(x)} \cdot 2 \int_x^m e^{-2C(\xi)} g(\xi)\, d\xi = x'^2.$$

Then

$$\frac{dx}{dt} = -\sqrt{e^{2C(x)} \cdot 2 \int_x^m e^{-2C(\xi)} g(\xi)\, d\xi}$$

and

$$T(m) = \int_0^m \frac{dx}{\sqrt{e^{2C(x)} \cdot 2 \int_x^m e^{-2C(\xi)} g(\xi)\, d\xi}}. \tag{14}$$

In view of the condition (11) there exists $\delta > 0$ such that $T(m) > \delta$ as $m \to \infty$.

Corollary 4.1 *Equation (10) does not possess the S-property in case of $c(x) = c = const$, if $g(x)$ satisfies the condition (A1) and*

$$|g(x)| \leq K\, e^{\kappa x} \text{ for large in modulus } x, \quad \kappa < 2c, \quad K > 0. \tag{15}$$

Proof: If the condition (15) fulfils the integral in (11) converges.

If function $g(x)$ in (10) grows rapidly and $c(x)$ grows comparatively slow then equation (10) possesses the S-property.

Theorem 4 *Suppose $g(x)$ in (10) satisfies the condition (A1). Equation (10) possesses the S-property if*

1) *solutions of (10) extend to the interval $[a, b]$;*

2) $|g(x)| \geq K\, e^{2c|x|}$ *for* $|x| > \Delta$, *where* $|c(x)| \leq c = const$ *for any* $x \in R, \quad K > 0.$

We need the following auxiliary result to prove the theorem.

Lemma 1 *A solution $x(t)$ of the Cauchy problem*

$$x'' + K\, e^{2cx} = c\, x'^2, \quad x(t_0) = m, \quad x'(t_0) = 0 \tag{16}$$

has a zero $t_1 > t_0$, if m is sufficiently large and

$$t_1(m) - t_0 \to 0 \text{ as } m \to +\infty. \tag{17}$$

Proof: It follows from (13) that

$$\frac{dz}{dx} = 2c \cdot z - 2Ke^{2cx}, \quad z(m) = 0, \tag{18}$$

where $z = x'^2$ and $x(t)$ is a solution of (16). One gets by integrating (18) that

$$z(x)e^{2cx} \cdot 2 \int_x^m e^{-2cs} \cdot Ke^{2cs}\, ds = e^{2cx} \cdot 2K(m - x) = x'^2.$$

Then

$$\frac{dx}{dt} = -\sqrt{e^{2cx} \cdot 2K(m - x)}$$

and

$$\int_x^m \frac{ds}{e^{cs} \cdot \sqrt{2K(m - x)}}.$$

We have then the expression

$$\frac{1}{\sqrt{2K}} \int_0^m \frac{e^{-cs} ds}{\sqrt{m - x}} = t_1(m) - t_0 \tag{19}$$

for the first zero t_1 of $x(t)$ in the interval (t_0, b).

By the variable changes $\frac{s}{m} = \xi$ and $1 - \xi = \eta$ the integral $\int_0^m \frac{e^{-cs} ds}{\sqrt{m-x}}$ is reduced to

$$\frac{e^{-cm}}{\sqrt{m}} \int_0^1 \frac{e^{cm\eta} ds}{\sqrt{\eta}}. \tag{20}$$

Bringing (19) and (20) together one obtains

$$\begin{aligned} t_1(m) - t_0 &= \frac{1}{\sqrt{2Km}} e^{-cm} \int_0^1 \frac{e^{cm\eta} ds}{\sqrt{\eta}} \\ &\leq \frac{1}{\sqrt{2Km}} e^{-cm} \cdot e^{cm} \int_0^1 \eta^{-\frac{1}{2}}\, d\eta = \sqrt{\frac{2}{Km}}. \end{aligned}$$

Lemma 2 *Let $g(x) \geq K\, e^{2cx}$ for $x \geq \Delta$.*

Then a solution $x(t)$ of the Cauchy problem

$$x'' + g(x) = c(x)\, x'^2, \quad x(t_0) = m, \quad x'(t_0) = 0, \tag{21}$$

where $|c(x)| \leq c \quad \forall x$, has a zero $t_1 > t_0$, if m is sufficiently large and

$$t_1(m) - t_0 \to 0 \text{ as } m \to +\infty. \tag{22}$$

Proof: By comparison of respective solutions $x(t)$ and $y(t)$ of the problems (21) and (16). We wish to show that given $\varepsilon > 0$ one can find m_0 such that $t_1(m) - t_0 < \varepsilon$ if $m > m_0$, where t_1 is the first zero of (t) in the interval (t_0, b).

a. First of all let us mention that extendability of solutions of (21) and Theorem 15.11 in [2] imply that there exists a function $\omega(\rho_0)$ such that $\omega(\rho_0) \to \infty$ and

$$x^2(t) + x'^2(t) > \omega^2(\rho_0) \text{ in the interval } [a, b] \tag{23}$$

if

$$x^2(t_0) + x'^2(t_0) > \rho_0^2. \tag{24}$$

b. One gets that in an interval $[t_0, t_\Delta]$ where $x(t) \geq \Delta$ (Δ is as in conditions of Theorem 4)

$$\begin{aligned} x''(t) + g(x(t)) &= x''(t) + K\, e^{2cx(t)} + \varepsilon_1(t) \\ &= c\, x'(t)^2 - \varepsilon_2(t), \end{aligned}$$

where $\varepsilon_1(t) \geq 0$ and $\varepsilon_2(t) \geq 0$. Finally,

$$x''(t) + K\, e^{2cx(t)} = c\, x'(t)^2 - \varepsilon(t), \tag{25}$$

where $\varepsilon(t) \geq 0$. Since $x'(t_0) = 0$, $x(t)$ is a decreasing function in some right neighbourhood of $t = t_0$. Compare solutions $x(t)$ and $y(t)$ of the equations (21) and (16) in the interval $[t_0, t_\Delta]$, where $x(t) \in [\Delta, m]$. Since $y(t)$ solves equation

$$y'' + K\, e^{2cy(t)} = c\, y'(t)^2, \tag{26}$$

one gets passing to the first order equations with respect to u and v that

$$\frac{dz_1}{dx} = 2cz_1 - 2Ke^{2cx} - 2\varepsilon(t(x)), \quad z_1(m) = 0, \tag{27}$$

$$\frac{dz_2}{dx} = 2cz_1 - 2Ke^{2cx}, \quad z_2(m) = 0, \tag{28}$$

where $z_1 = x'^2$, $z_2 = y'^2$. One has that

$$z_1(x) \geq z_2(x), \quad x \in [\Delta, x_0). \tag{29}$$

Then

$$\left|\frac{dx}{dt}(t_x)\right| \geq \left|\frac{dy}{dt}(t_y)\right|, \tag{30}$$

where $x(t_x) = y(t_y) \in [\Delta, x_0)$. Let m_1 be so large that $\tau_1 - t_0 < \frac{\varepsilon}{2}$, where τ_1 is such that $y(\tau_1) = \Delta$, $\Delta < y(t) < m$ for $t \in (t_0, \tau_1)$. Then, by virtue of (30),

$$t_\Delta - t_0 \leq \tau_1 - t_0 < \frac{\varepsilon}{2}, \tag{31}$$

where t_Δ is such that $x(t_\Delta) = \Delta$, $\Delta < x(t) < m$ for $t \in (t_0, \tau_1)$.

c. Let m_1 be so large that $x'^2(t) > M_1^2$ for $t \in (t_\Delta, t_1]$, where M_1 is such that

$$\Delta = |x(t_1) - x(t_\Delta)| = \left| \int_{t_\Delta}^{t_1} x'(s)\, ds \right| \geq M_1 |t_1 - t_\Delta|$$

and

$$\frac{\varepsilon}{2} \geq \frac{\Delta}{M_1} \geq t_1 - t_\Delta. \tag{32}$$

d. Take $m_0 = \max\{m_1, m_2\}$. Then respective solutions $x(t)$ of (21), where $m > m_0$, are such that they have a zero t_1 in the interval (t_0, b) and, by virtue of (31) and (32),

$$t_1 - t_0 = t_1 - t_\Delta + t_\Delta - t_0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \tag{33}$$

q.e.d.

Lemma 3 *Let conditions of Lemma 2 hold.*

Then given $\varepsilon > 0$ there exists Λ such that any solution of the equation in (21) with the initial values

$$x(t_0) = x_0, \quad x'(t_0) = y_0, \quad t_0 \in [a, b), \tag{34}$$

sush that $x_0^2 + y_0^2 > \Delta^2$, has a zero $t_1 \in (t_0, t_0 + \varepsilon)$.

Proof: By the same type arguments as that of Lemma 4.2 in [5].

Proof of the theorem. Let an integer N be given. Define $\delta = \frac{b-a}{N+1}$.

Let Δ_1 be such that a solution of (10) with $x^2(a) + x'^2(a) > \Delta_1$ vanishes at some point t_1 to the right of a and $t_1 - a < \delta$. This is possible by Lemma 3.

Let Δ_2 be such that a solution of (10) with $x^2(a) + x'^2(a) > \Delta_2$ changes from an extremal point x_0 to zero in time less than a quarter of δ. This is possible by Lemma 2.

Finally, let Δ_3 be such that any solution of (10) with $x^2(a) + x'^2(a) > \Delta_3$ satisfies the inequality

$$x^2(t) + x'^2(t) > \max\{\Delta_1, \Delta_2\}$$

in $[a, b]$. This is possible in view of extendability of solutions of (10).

Then the distance between consequtive zeroes of any solution of (10) is less than δ and the total number of zeros is greater than N, q.e.d.

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Ю.А. Клоков, Ф.Ж. Садырбаев. Дифференциальные уравнения с экспоненциально растущими нелинейностями.

Аннотация. Рассматриваются дифференциальные уравнения вида $x'' + g(x) = f(t, x, x')$, где $g(x)$ экспоненциально растущая функция, а f "медленно растущая" функция. Приводятся условия на g и f, обеспечивающие суперлинейное поведение решений. Например, задача Дирихле для таких уравнений имеет бесконечно много решений.

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Ju.A. Klokovs, F. Sadirbajevs. Diferenciālvienādojumi ar eksponenciāli augošām nelinearitātēm.

Anotācija. Tiek aplūkoti diferenciālvienādojumi formā $x'' + g(x) = f(t, x, x')$, kur $g(x)$ ir eksponenciāli augoša funkcija un f ir "lēni" augoša funkcija. Doti nosacījumi, kuri nodrošina atrisinājumu superlineāro uzvedību. Piemēram, Dirihlē problēmai ir bezgalīgi daudz atrisinājumu.

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