Lemniscatic functions in the theory of the Emden - Fowler differential equation

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Summary. We investigate functions which are solutions of the equation $x'' = -2x^3$ and related ones. Solutions which satisfy also the initial conditions $x(0) = 0, x'(0) = 1$ and $x(0) = 1, x'(0) = 0$ are known as lemniscatic sine and cosine functions respectively. Taking our cue from the theory of elementary trigonometric functions, we give our own proof of the most remarkable properties and provide various formulae for relations between lemniscatic functions and their derivatives. Everywhere, if possible, the analogies with trigonometric functions are shown. Our main tool is the theory of Jacobi elliptic functions.

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1 Introduction

In this paper we consider non-elementary functions which arise when considering the Emden - Fowler equation

$$x'' = -2x^3. \tag{1}$$

We are motivated by various definitions of the classical $\sin t$ function and rich theory of elementary trigonometric functions. It is known ([11][Ch. 1, §5], for example) that properties of $\sin t$ and $\cos t$ can be derived directly from the differential equation

$$x'' = -x. \tag{2}$$

Let us recall four definitions of $\sin t$.

First definition. Classical definition using the unit circle.

Second definition. The function $\sin t$ is defined as a solution of the Cauchy problem

$$x'' = -x, \quad x(0) = 0, \quad x'(0) = 1. \tag{3}$$

Third definition. The function $\sin t$ is introduced as the upper limit in

$$t = \int_0^x \frac{ds}{\sqrt{1 - s^2}}$$

and then it is extended to the whole R by symmetry and periodicity.

Fourth definition introduces $\sin t$ as a sum of infinite series

$$t - \frac{t^3}{3!} + \ldots$$

In what follows we consider a nonlinear equation (1) and discuss properties of its solutions and related functions. It appears that the theory of non-elementary functions related to equation (1) is in some respects similar to that for the linear equation (2). We believe that formulae obtained can be applied also for investigation of non-autonomous equations of Emden - Fowler type.

2 Non-elementary sine function

We start with **the second definition** and introduce the function $S(t)$ as a solution to equation (1) subject to the initial conditions

$$x(0) = 0, \quad x'(0) = 1. \tag{4}$$

Standard considerations show that this function is a sin-like function with a minimal period $4A$, where

$$A = \int_0^1 \frac{ds}{\sqrt{1 - s^4}} \tag{5}$$

and the value of A is approximately 1.311...

The phase trajectory of this solution is a closed curve

$$x'^2(t) + x^4(t) = 1. \tag{6}$$

The simple form of the expression above is due to the coefficient 2 in the right side of (1).

Third definition. Function $S(t)$ can be defined also as

$$t = \int_0^{S(t)} \frac{ds}{\sqrt{1 - s^4}} \tag{7}$$

on the interval $[0, A]$ and then extended to the whole R in a *sin*-like manner. Obviously, zeros of this function are $t_i = 2iA$ and points of extrema are $(2i+1)\,A$ $(i = 0, \pm 1, \pm 2, \ldots)$.

Fourth definition. It is difficult task to calculate sequential derivatives of $S(t)$ at the point $t = 0$ in order to construct a series expansion. Instead we use the method of successive approximations as shown in the book [11].

Consider the Cauchy problem (1), (4), and construct successive approximations to the function S. Rewrite equation (1) in the form of a system of two equations

$$u' = v, \quad v' = -2u^3, \tag{8}$$

and consider it with the initial conditions

$$u(0) = 0, \quad v(0) = 1, \tag{9}$$

where $u = x$, $v = x'$.

We use the following formulae

$$\begin{cases} u_0(x) \equiv 0, \quad v_0(x) \equiv 1; \\ u_n(x) = \int\limits_0^x v_{n-1}(t)\, dt \quad (n = 1, 2, \dots)\ , \\ v_n(x) = 1 - 2\int\limits_0^x u_{n-1}^3(t)\, dt \quad (n = 1, 2, \dots) \end{cases}$$

to compute successive approximations.

Computations performed in ([9]) show that

$$u_5(t) = t - \frac{t^5}{10} + \frac{t^9}{120} - \frac{t^{13}}{2600} + \frac{t^{17}}{136000} - \dots \tag{10}$$

As a byproduct iterations for $S'(t)$ are obtained:

$$v_5(t) = 1 - \frac{t^4}{2} + \frac{3t^8}{40} - \frac{x^{12}}{150} + \frac{x^{16}}{8000} - \dots . \tag{11}$$

We obtained thus the following recurrent formula for u_n :

$$u_{2i+1} = \int_0^t \left[1 - 2\int_0^s u_{2i-1}^3(\xi)\, d\xi\right] ds, \quad i = 1, 2, \dots, \quad u_1 \equiv 1.$$

We mention without proof several properties of the function $S(t)$, which can be obtained easily by integration of the equation (1), taking into account its autonomity.

1° *Function $S(t)$ is periodic with the minimal period $4A$, where A is given in (5).*

2° *$S(t)$ is an odd function.*

3° *$S(t)$ monotonically increases in the interval $[-A, A]$ and*

$$\forall t \in [-A; A]: \ t = \int_0^{S(t)} \frac{ds}{\sqrt{1 - s^4}}. \tag{12}$$

4° *$S(t)$ takes its maximal values $+1$ at the points $(4i + 1)A$ and minimal values -1 at the points $(4i - 1)A$, $(i = 0, \pm 1, \pm 2, \dots)$, where $S'(t) = 0$.*

Historical note. The integral in (7) appears in the theory of the lemniscate curve given by $(x^2 + y^2)^2 = x^2 - y^2$. The function $S(t)$ is known as *a lemniscatic sine* and denoted as *sin lemn* t in [12, § 22.8] or *sl* t, the notation first introduced by K. Gauss.

3 Related functions

The function $C(t)$ can be introduced as a solution to the Cauchy problem

$$x'' = -2x^3, \quad x(0) = 1, \quad x'(0) = 0. \tag{13}$$

In some respect $C(t)$ is an analogue of $\cos t$ function. By properties of autonomous equations $C(t) = S(t + const)$ for any $t \in R$. It follows from the initial conditions in (13) that A is appropriate constant. So $C(t) = S(t + A)$. Evidently $C(t)$ is an even function with a minimal period $4A$, with the value range $[-1, 1]$.

The reduction formulae are valid for $S(t)$ and $C(t)$, which follow from the basic relation $C(t) = S(t+A)$, periodicity and evenness and oddness properties. The reduction formulae are the same as those for sin and cos with evident changes (the constant A is an analogue of $\pi/2$).

Remark 3.1. The function $C(t)$ is known as *a lemniscatic cosine* and denoted by *coslemnt* in [12, § 22.8] or *cl t* by some other authors, starting from K. Gauss.

Remark 3.2. A non-elementary tangent function can be introduced by $T(t) = \frac{S(t)}{C(t)}$. It is periodic and the period is $2A$. The images are not defined in the points $A + 2Ak$, k an integer. The range is R.

In the same manner analogues of $\cot t$, $\sec t$, $\csc t$, $\arcsin x$, $\arccos x$ and $\arctan x$ can be introduced and investigated. In what follows we draw our attention mainly to the analogue of $\arcsin x$ function. We denote this function $F(x)$ and provide detailed discussion about its properties.

4 Inverse functions

The function $F(x)$ is introduced as an inverse function to $S(t)$. Standard computation shows that $F(x)$ is a solution to the Cauchy problem

$$w'' = 2x^3 \left(w'\right)^3, \tag{14}$$

$$w(0) = 0, \;\; w'(0) = 1. \tag{15}$$

This problem has explicit solution

$$w = \int_0^x \frac{ds}{\sqrt{1-s^4}}.$$

The following properties of the function $F(x)$ are direct consequences of its definition and related properties of the function $S(t)$:

1° *Function $F(x)$ is defined for any $x \in (-1; 1)$.*

2° *$F(x)$ is an odd function.*

3° *$F(x)$ is differentiable (hence continuous) for $x \in (-1;1)$ and*

$$\forall x \in (-1;1): \; F'(x) = \frac{1}{\sqrt{1-x^4}}\,. \tag{16}$$

4° *$F(x)$ is monotone increasing function for $x \in (-1;1)$.*

5° *$F(x)$ is concave for $x \in (-1;0)$ and convex for $x \in (0;1)$.*

6° *There exist the one-sided limits $\lim_{x\to 1-} F(x)$ and $\lim_{x\to -1+} F(x)$.*

Successive approximations can be computed for the function F considering the system

$$\begin{cases} u' &= \; v, \\ v' &= \; 2x^3v^3 \\ u(0)=0, & \; v(0)=1. \end{cases}$$

For example

$$u_3(x) = x + \frac{x^5}{10} + \frac{x^9}{24} + \frac{x^{13}}{104} + \frac{x^{17}}{1088}.$$

Instead we provide the Taylor series expansion for $F(x)$ in Section 8.

Remark 4.1. The following Taylor series expansions are known for related functions ([3, 15. lpp.]):

$$F'(x)\,F(x) = x + \sum_{n=1}^{\infty} \frac{3\cdot 7\cdot \cdots \cdot (4n-1)}{5\cdot 9\cdot \cdots \cdot (4n+1)} x^{4n+1},$$

$$\left[F(x)\right]^2 = x^2 + \sum_{n=1}^{\infty} \frac{3\cdot 7\cdot \cdots \cdot (4n-1)}{5\cdot 9\cdot \cdots \cdot (4n+1)} \cdots \frac{x^{4n+2}}{2n+1}.$$

5 General solution

Trajectories

$$x'^2 + x^4 = r^4 \tag{17}$$

of equation (1) fill the phase plane if the parameter r varies from zero to infinity. Fix the r constant for a moment. Any solution of (1) with the trajectory $(x(t), x'(t))$ given by (17) satisfies the initial conditions

$$x(0) = \alpha, \quad x'(0) = \beta \tag{18}$$

for some α and β, where $\beta^2 + \alpha^4 = r^4$. Since the trajectory associated with the solution of (1) with the initial conditions

$$x(0) = 0, \quad x'(0) = r, \quad r > 0, \tag{19}$$

also lies on the curve (17), any solution of (1) with the initial conditions (18) is obtained from the solution $S_r(t)$ of (1), (19) by shift along the t-axis. Thus any solution of (1), (18) has the form

$$x(t) = S_r\,(t + const). \tag{20}$$

Evidently $S_r(t) \equiv S(t)$ for $r = 1$. Easy computation shows that

$$S_r(t) = r \cdot S(r\,t). \tag{21}$$

The above arguments can be summarized in the following statement.

Proposition 5.1 *Any solution of the equation (1) can be obtained from the formula*

$$x(t) = r \cdot S(r\,t + \varphi). \tag{22}$$

Remark 5.1. The above formula can be considered as a nonlinear analogue of the general solution in the form $x(t) = r\,\sin(t+\varphi)$ of the linear equation $x'' = -x$. A solution of the Cauchy problem (1), (18) can be found from (5.1) by solving the system

$$r \cdot S(\varphi) = \alpha,$$
$$r^2 \cdot S'(\varphi) = \beta.$$

Then $r = (\beta^2 + \alpha^4)^{1/4}$ and φ is to be determined from the system

$$S(\varphi) = \frac{\alpha}{(\beta^2 + \alpha^4)^{1/4}}, \quad \operatorname{sign} \mathrm{S}'(\varphi) = \operatorname{sign}\beta.$$

For example, a solution to the problem

$$x'' = -2x^3, \quad x(0) = k, \quad x'(0) = 0. \tag{23}$$

is given by the formula

$$x(t) = k \cdot S(k\,t + A),$$

where A is given in (5).

Remark 5.2. There is alternative formula for the general solution of $x'' = -x$, namely, $x(t) = C_1\,\sin t + C_2\,\cos t$. It can be written also in the form

$$x(t) = \sqrt{C_1^2 + C_2^2} \cdot \left(\frac{C_1}{\sqrt{C_1^2 + C_2^2}}\,\sin t + \frac{C_2}{\sqrt{C_1^2 + C_2^2}}\,\cos t \right). \tag{24}$$

Using the sum formula (52) one obtains the alternative expression for the general solution of $x'' = -2\,x^3$

$$x(t) = r \cdot \frac{S(rt)C(\varphi) + S(\varphi)C(rt)}{1 - S(rt)C(rt)S(\varphi)C(\varphi)},$$

which is a nonlinear analogue of (24).

6 Jacobi elliptic functions. Preliminaries

Jacobi elliptic functions will be used in the next section to derive several useful formulae for the functions S_3 and C_3. Let us consider the system

$$\begin{aligned} x_1' &= x_2 x_3, \\ x_2' &= -x_1 x_3, \\ x_1' &= -k^2 x_1 x_2, \end{aligned} \tag{25}$$

together with the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = 1. \tag{26}$$

The coefficient k is supposed to satisfy $0 < k^2 < 1$.

The main three Jacobi elliptic functions (for k given) can be introduced as

$$\operatorname{sn} t = x_1(t), \quad \operatorname{cn} t = x_2(t), \quad \operatorname{dn} t = x_3(t).$$

Other nine Jacobi elliptic functions can be introduced as some ratios involving the functions above. In what follows we use also the function

$$\operatorname{sd} t = \frac{\operatorname{sn} t}{\operatorname{dn} t}. \tag{27}$$

It is mentioned in [1][Ch. 16, § 22] that formulae of general terms in Taylor series expansions for the main Jacobi elliptic functions are unknown but the first terms are (see also [11][Ch. 1, § 6]) :

$$\begin{aligned} \operatorname{sn} t &= t - (1+k^2)\tfrac{t^3}{3!} + (1+14k^2+k^4)\tfrac{t^5}{5!} + \ldots \\ \operatorname{cn} t &= 1 - \tfrac{t^2}{2!} + (1+4k^2)\tfrac{t^4}{4!} - (1+44k^2+16k^4)\tfrac{t^6}{6!} + \ldots \\ \operatorname{dn} t &= 1 - k^2\tfrac{t^2}{2!} + k^2(4+k^2)\tfrac{t^4}{4!} - k^2(16+44k^2+k^4)\tfrac{t^6}{6!} + \ldots \end{aligned} \tag{28}$$

The parameter k is fixed. The expansion series above are known to have the finite radius of convergence K, where

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}}. \tag{29}$$

The integral above converges as $k^2 < 1$ and diverges as $k^2 = 1$. The value of K is approximately 1.8540746773.

Elliptic functions of Jacobi satisfy the relations ([5, Ch. VII, § 1]):

$$\operatorname{sn}^2 t + \operatorname{cn}^2 t = 1, \quad k^2 \operatorname{sn}^2 t + \operatorname{dn}^2 t = 1, \tag{30}$$

which, in turn, imply that

$$\operatorname{dn}^2 t - k^2 \operatorname{cn}^2 t = 1 - k^2. \tag{31}$$

Thus $\operatorname{sn} t$ and $\operatorname{cn} t$ behave like usual $\sin\tau$ and $\cos\tau$ functions, while $\operatorname{dn}\tau$ oscillates between $1-k^2$ and 1. The function dn is positive if $0 < k^2 < 1$. Moreover, the function $\operatorname{sn} t$ is odd, $\operatorname{cn} t$ and $\operatorname{dn} t$ are even functions.

It follows from (25) that

$$\frac{dt}{dx_1} = \frac{1}{x_2 x_3} = \frac{1}{\sqrt{(1-x_1^2)(1-k^2x_1^2)}}. \tag{32}$$

One has then that

$$x_1(K) = \operatorname{sn} K = 1, \quad x_2(K) = \operatorname{cn} K = 0, \quad x_3(K) = \operatorname{dn} K = k, \tag{33}$$

where K is given in (29). So $\operatorname{sn} t$ and $\operatorname{cn} t$ are periodic functions with a minimal period of $4K$. The function $\operatorname{dn} t$ is periodic with a minimal period $2K$.

It follows from (32) that

$$t = \int_0^{x_1} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}. \tag{34}$$

and thus $x_1 = \operatorname{sn} t$ is the inverse function for the incomplete Legendre elliptic integral of the first kind.

The following formulae ([8, 753.-765. lpp.], [5, Ch. VII, § 6]) will be used below to derive the analogous ones for the functions S and C (it is supposed that a constant k is fixed):

$$\operatorname{sn}(-t) = -\operatorname{sn} t, \quad cn(-t) = cnt, \quad dn(-t) = dnt; \tag{35}$$

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}; \tag{36}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}; \tag{37}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{38}$$

$$\operatorname{sn}(t+K) = \frac{\operatorname{cn} t}{\operatorname{dn} t}, \tag{39}$$

$$\operatorname{cn}(t+K) = -\sqrt{1-k^2}\frac{\operatorname{sn} t}{\operatorname{dn} t}, \tag{40}$$

$$\operatorname{dn}(t+K) = \sqrt{1-k^2}\frac{1}{\operatorname{dn} t}, \tag{41}$$

$$\operatorname{sn}(t+2K) = -\operatorname{sn} t, \quad \operatorname{cn}(t+2K) = -\operatorname{cn} t. \tag{42}$$

The periodicity of the elliptic functions can be written as

$$\operatorname{sn}(t+4K) = \operatorname{sn} t, \quad \operatorname{cn}(t+4K) = \operatorname{cn} t, \quad \operatorname{dn}(t+2K) = \operatorname{dn} t.$$

Relations for derivatives:

$$(\operatorname{sn} x)' = \operatorname{cn} x \operatorname{dn} x, \quad (\operatorname{cn} x)' = -\operatorname{sn} x \operatorname{dn} x, \quad (\operatorname{dn} x)' = -k^2 \operatorname{sn} x \operatorname{cn} x. \tag{43}$$

Remark 6.1. For $k^2 = 1$

$$\operatorname{sn} t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad \operatorname{cn} t = \operatorname{dn} t = \frac{2}{e^t + e^{-t}};$$

for $k = 0$

$$\operatorname{sn} t = \sin t, \quad \operatorname{cn} t = \cos t, \quad \operatorname{dn} t \equiv 1.$$

So the functions sn, cn and dn are not periodical yet for $k = 1$.

7 Basic relations involving $S(t)$ and $C(t)$ functions

In some respect relations between S and C resemble the relations between $\sin t$ and $\cos t$. In what follows we derive some formulae for $S(t)$, $C(t)$, $S'(t)$, $C'(t)$ functions of which the main is the sum formula for $S(\alpha + \beta)$. Our main tool is the theory of Jacobi elliptic functions.

Denote

$$J(x) := \int_0^x \frac{ds}{\sqrt{(1 - s^2)}\sqrt{(1 - k^2 s^2)}},$$

where $k := \frac{1}{\sqrt{2}}$.

Proposition 7.1

$$F(x) = k \operatorname{sign} \mathrm{x} \left(\mathrm{K} - \mathrm{J}(\sqrt{1 - \mathrm{x}^2})\right), \quad -1 \le \mathrm{x} \le 1, \tag{44}$$

where $K = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$ *and* $F(x) = \int\limits_0^x \frac{dt}{\sqrt{(1-t^4)}}$.

Proof. Notice that $A = k\,K$, where K is defined in (29). Indeed,

$$\begin{aligned} A = & \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1+t^2)}} = \left(\begin{array}{c} \text{substitution} \\ t^2 + s^2 = 1 \end{array} \right) \\ = & -\int\limits_1^0 \frac{ds}{\sqrt{(1-s^2)(2-s^2)}} = k \cdot \int\limits_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}}. \end{aligned} \tag{45}$$

Consider the function $f(x) = F\left(\sqrt{1 - x^2}\right)$ in the interval $[0, 1)$. Since

$$\frac{df}{dx} = -\frac{1}{\sqrt{(2 - x^2)(1 - x^2)}}$$

and

$$f(0) = F(1) = A$$

one obtains for $0 \le z < 1$ that

$$f(x) - f(0) = F(\sqrt{1 - x^2}) - A = -\int\limits_0^x \frac{ds}{\sqrt{(2 - s^2)(1 - s^2)}}$$

and hence

$$F\left(\sqrt{1-x^2}\right) = A - k \cdot J(x) = k \cdot (J(1) - J(x)), \quad x \in [0,1]. \tag{46}$$

Then

$$F(z) = k \cdot (J(1) - J(\sqrt{1-z^2}), \quad 0 \le z \le 1.$$

Since $F(x)$ is odd function and $J(1) = K$ one has

$$F(x) = k \,\mathrm{sign\,x} \left(\mathrm{K} - \mathrm{J}(\sqrt{1-\mathrm{x}^2})\right), \quad -1 \le \mathrm{x} \le 1,$$

q.e.d.

Corollary 7.1

$$S(t) = \mathrm{cn}\left(K - \frac{t}{k}\right) = k \cdot \mathrm{sd}\left(\frac{t}{k}\right). \tag{47}$$

Proof. Since S and F are mutually inverse functions one has that

$$S(F(x)) = S\left(k\left[K - J(\sqrt{1-x^2})\right]\right) = x$$

or

$$S(t) = x, \quad -1 \le x \le 1,$$

where $t = k\left[K - J\left(\sqrt{1-x^2}\right)\right]$, $K(k)$ is given in (29) and $k = \frac{1}{\sqrt{2}}$. Evidently $S(-A) = -1$, $S(A) = 1$ and $-A \le t \le A$ as $-1 \le x \le 1$. We have that

$$K - \frac{t}{k} = J(\sqrt{1-x^2}), \quad -1 \le x \le 1$$

and thus by definition of sn

$$\mathrm{sn}(K - \frac{t}{k}) = \mathrm{sn}(J(\sqrt{1-x^2})) = \sqrt{1-x^2}.$$

From this $x = \sqrt{1 - \mathrm{sn}^2(K - \frac{t}{k})} = \mathrm{cn}(K - \frac{t}{k})$ and finally $S(t) = \mathrm{cn}(K - \frac{t}{k})$, q.e.d.

Corollary 7.2

$$C(t) = \mathrm{cn}\left(\frac{t}{k}\right). \tag{48}$$

Proof. Notice that $A = k\,K$. One has then using (47)

$$C(t) = S(t+A) = \mathrm{cn}\left(K - \frac{t+A}{k}\right) = \mathrm{cn}\left(-\frac{t}{k}\right) = \mathrm{cn}\left(\frac{t}{k}\right).$$

Remark 7.1. The relations $S(t) = k \cdot \mathrm{sd}(\frac{t}{k}) = 2^{-\frac{1}{2}} \cdot \frac{\mathrm{sn}(\sqrt{2}\,t)}{\mathrm{dn}(\sqrt{2}\,t)}$ and $C(t) = \mathrm{cn}(\frac{t}{k}) = \mathrm{cn}(\sqrt{2}\,t)$ are known (see, for example, [12, § 22.8]).

Remark 7.2. Let us give alternative proof of the relation $C(t) = \operatorname{cn}\left(\frac{t}{k}\right)$, which do not use the integrals. Consider the system (25), (26). We will show that the function $x_2(t) = \operatorname{cn} t$ satisfies the equation

$$x_2'' = -2\,k^2\,x_2^3. \tag{49}$$

Indeed, it follows from (25) that

$$x_2'' = (-x_1\,x_3)' = -(x_1'x_3 + x_1x_3') = -(x_2x_3^2 + x_1(-k^2x_1x_2)) = k^2x_1^2x_2 - x_2x_3^2. \tag{50}$$

It follows from (30) and (31) and $1 - k^2 = k^2$ that

$$x_1^2 = 1 - x_2^2, \quad x_3^2 = k^2x_2^2 + (1 - k^2) = k^2x_2^2 + k^2 \tag{51}$$

and (51) continues as

$$x_2'' = k^2(1 - x_2^2)x_2 - x_2(k^2x_2^2 + k^2) = -2\,k^2\,x_2^3.$$

The function x_2 satisfies also

$$x_2(0) = 1,\;\; x_2'(0) = -x_1(0)x_3(0) = 0.$$

Then the function $\operatorname{cn}\left(\frac{t}{k}\right) = x_2\left(\frac{t}{k}\right)$ is a solution of the Cauchy problem

$$x'' = -2\,x^3, \quad x(0) = 1,\;\; x'(0) = 0,$$

q.e.d.

Proposition 7.2

$$S(\alpha+\beta) = \frac{S(\alpha)C(\beta) + S(\beta)C(\alpha)}{1 - S(\alpha)S(\beta)C(\alpha)C(\beta)}. \tag{52}$$

Proof. One obtains by using formulae (36) to (42)that

$$\begin{aligned} S(\alpha+\beta) &= \frac{k\operatorname{sn}\left(\frac{\alpha}{k}+\frac{\beta}{k}\right)}{\operatorname{dn}\left(\frac{\alpha}{k}+\frac{\beta}{k}\right)} = \frac{k\operatorname{sn}\frac{\alpha}{k}\operatorname{cn}\frac{\beta}{k}\operatorname{dn}\frac{\beta}{k} + k\operatorname{sn}\frac{\beta}{k}\operatorname{cn}\frac{\alpha}{k}\operatorname{dn}\frac{\alpha}{k}}{\operatorname{dn}\frac{\alpha}{k}\operatorname{dn}\frac{\beta}{k} - k^2\operatorname{sn}\frac{\alpha}{k}\operatorname{sn}\frac{\alpha}{k}\operatorname{sn}\frac{\beta}{k}\operatorname{cn}\frac{\beta}{k}} \\ &= \frac{\frac{k\operatorname{sn}\frac{\alpha}{k}\operatorname{cn}\frac{\beta}{k}\operatorname{dn}\frac{\beta}{k}}{\operatorname{dn}\frac{\alpha}{k}\operatorname{dn}\frac{\beta}{k}} + \frac{k\operatorname{sn}\frac{\beta}{k}\operatorname{cn}\frac{\alpha}{k}\operatorname{dn}\frac{\alpha}{k}}{\operatorname{dn}\frac{\alpha}{k}\operatorname{dn}\frac{\beta}{k}}}{1 - \frac{k^2\operatorname{sn}\frac{\alpha}{k}\operatorname{cn}\frac{\alpha}{k}\operatorname{sn}\frac{\beta}{k}\operatorname{cn}\frac{\beta}{k}}{\operatorname{dn}\frac{\alpha}{k}\operatorname{dn}\frac{\beta}{k}}} = \frac{\frac{k\operatorname{sn}\frac{\alpha}{k}}{\operatorname{dn}\frac{\alpha}{k}}\operatorname{cn}\frac{\beta}{k} + \frac{k\operatorname{sn}\frac{\beta}{k}}{\operatorname{dn}\frac{\beta}{k}}\operatorname{cn}\frac{\alpha}{k}}{1 - \frac{k\operatorname{sn}\frac{\alpha}{k}}{\operatorname{dn}\frac{\alpha}{k}}\frac{k\operatorname{sn}\frac{\beta}{k}}{\operatorname{dn}\frac{\beta}{k}}\operatorname{cn}\frac{\alpha}{k}\operatorname{cn}\frac{\beta}{k}} \\ &= \frac{S(\alpha)C(\beta) + S(\beta)C(\alpha)}{1 - S(\alpha)S(\beta)C(\alpha)C(\beta)}. \end{aligned}$$

Remark 7.3. The above formula is an analogue of the usual expression for the $\sin(\alpha+\beta)$.

Remark 7.4. Investigations of the sum formula for $S(t)$ go back to Fagnano and L. Euler ([6, §§ 2.1, 2.2, 2.3]). The sum formula was known rather in the form

$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1 + u^2v^2}, \tag{53}$$

where $u = S(\alpha)$, $v = S(\beta)$, $r = S(\alpha+\beta)$. This form is applicable in some vicinity of zero. It can be shown that (52) and (53) are equivalent for sufficiently small values of arguments.

Corollary 7.3

$$S(t + A) = C(t) \textit{ and } C(t + A) = -S(t). \tag{54}$$

Proof. By direct computation for the first equality and using the relation $C(t+A) = S(t + 2A)$ for the second one.

Remark 7.5. The above assertion is also a consequence of the autonomity of equation in (13). Various reduction formulae can be derived for the functions S and C just like in the case of elementary functions $\sin t$ and $\cos t$. A constant A will appear everywhere instead of $\pi/2$. In the sequel we shall use the reduction formulae without making reference.

Corollary 7.4

$$S(2\alpha) = \frac{2S(\alpha)C(\beta)}{1 - S^2(\alpha)C^2(\alpha)}. \tag{55}$$

Proposition 7.3

$$C(\alpha+\beta) = \frac{C(\beta)C(\alpha) - S(\alpha)S(\beta)}{1 + S(\alpha)S(\beta)C(\alpha)C(\beta)}. \tag{56}$$

Proof. The relations (52) and (54) imply

$$C(\alpha+\beta) = C\big(\alpha + (\beta + A)\big) = \frac{S(\alpha)C(\beta+A) + S(\beta+A)C(\alpha)}{1 - S(\alpha)S(\beta+A)C(\alpha)C(\beta+A)} =$$
$$= \frac{-S(\alpha)S(\beta) + C(\beta)C(\alpha)}{1 + S(\alpha)C(\beta)C(\alpha)S(\beta)} = \frac{C(\beta)C(\alpha) - S(\alpha)S(\beta)}{1 + S(\alpha)S(\beta)C(\alpha)C(\beta)}.$$

Corollary 7.5

$$C(2\alpha) = \frac{C^2(\alpha) - S^2(\alpha)}{1 + S^2(\alpha)C^2(\alpha)}. \tag{57}$$

Proposition 7.4

$$S'(t) = C(t)(1 + S^2(t)). \tag{58}$$

Proof. Set $k = \frac{1}{\sqrt{2}}$ and notice that $\sqrt{1-k^2} = k$. The relations (47), (26) and (43) together imply

$$S'(x) = \mathrm{cn}'(-\frac{t}{k} + K) = \left(\frac{k\,\mathrm{sn}\,\frac{x}{k}}{\mathrm{dn}\,\frac{x}{k}}\right)' = \frac{k\left(\mathrm{sn}\,\frac{x}{k}\right)'\mathrm{dn}\,\frac{x}{k} - k\,\mathrm{sn}\,\frac{x}{k}\left(\mathrm{dn}\,\frac{x}{k}\right)'}{\mathrm{dn}^2\,\frac{x}{k}} =$$
$$= \frac{k\left(\mathrm{cn}\,\frac{x}{k}\right)\left(\mathrm{dn}\,\frac{x}{k}\right)\left(\frac{1}{k}\right)\left(\mathrm{dn}\,\frac{x}{k}\right) - k\left(\mathrm{sn}\,\frac{x}{k}\right)(-k^2)\left(\mathrm{sn}\,\frac{x}{k}\right)\left(\mathrm{cn}\,\frac{x}{k}\right)\left(\frac{1}{k}\right)}{\mathrm{dn}^2\,\frac{x}{k}} =$$
$$= \frac{\mathrm{cn}\,\frac{x}{k}\,\mathrm{dn}^2\,\frac{x}{k} + k^2\,\mathrm{sn}^2\,\frac{x}{k}\,\mathrm{cn}\,\frac{x}{k}}{\mathrm{dn}^2\,\frac{x}{k}} = C_3(x) + \frac{k^2\,\mathrm{sn}^2\,\frac{x}{k}}{\mathrm{dn}^2\,\frac{x}{k}}\cdot\mathrm{cn}\,\frac{x}{k} = C_3(x) + S^2(x)\cdot C(x) =$$
$$= C(x)\left(1 + S^2(x)\right).$$

Proposition 7.5

$$C'(t) = -S(t)(1 + C^2(t)). \tag{59}$$

Proof.

$$C'(t) = S'(t+A) = C'(t+A)\left(1 + S^2(t+A)\right) = -S(t)(1 + C^2(t)).$$

Remark 7.6. Formulae (58) and (59) are nonlinear analogues of $(\sin t)' = \cos t$ and $(\cos t)' = -\sin t$.

Proposition 7.6 *Any two of the relations*

$$(\mathbf{A})\, S'^2 + S^4 = 1, \quad (\mathbf{B})\, S'^2 = C^2(1+S^2)^2, \quad (\mathbf{C})\, S^2 + S^2C^2 + C^2 = 1$$

imply the remainder.

Proof. First, show that the implication (**A**), (**B**) $\to$ (**C**) is true. It follows from

$$S'^2 = 1 - S^4 = (1 - S^2)(1 + S^2)$$

and

$$S'^2 = C^2(1+S^2)(1+S^2)$$

that $(1 - S^2) = C^2(1 + S^2)$ and, finally

$$S^2(t) + S^2(t)C^2(t) + C^2(t) \equiv 1. \tag{60}$$

Implication (**A**), (**C**) $\to$ (**B**). Since

$$S^4 + S^4C^2 + S^2C^2 = S^2 \tag{61}$$

one has

$$S'^2 + (S^2 - S^4C^2 - S^2C^2) = 1 = S^2 + S^2C^2 + C^2$$

and

$$S'^2 = S^2C^2 + C^2 + S^4C^2 + S^2C^2 = C^2(1+S^2)^2.$$

Implication (**B**), (**C**) $\to$ (**A**). We have

$$S'^2 = C^2(1+S^2)^2 = S^2C^2 + C^2 + S^4C^2 + S^2C^2 - S^2 + S^2 = 1 + S^4C^2 + S^2C^2 - S^2$$

and by virtue of (61)

$$S'^2 + S^2 - S^4C^2 - S^2C^2 = S'^2 + S^4 = 1.$$

Remark 7.7. The identity (60) is known and can be found for instance in the form $S^2 = \frac{1-C^2}{1+C^2}$ in [12, Ex. 8, § 22.8]. It provides a formula for the “nonlinear unit circle". It can be written also as $(1 + S^2(t))(1 + C^2(t)) = 2$, which is simpler than $(1 + \sin^2 t)(1 + \cos^2 t) = 2 + 2\sin^2 t \cos^2 t$.

8 Series expansions

In the case under consideration the parameter k of Jacobi elliptic functions is $k = \frac{1}{\sqrt{2}}$. Using (28), one obtains

$$\begin{aligned} \operatorname{sn} z &= z - \left(1 + \frac{1}{2}\right)\frac{z^3}{3!} + \left(1 + 14 \cdot \frac{1}{2} + \frac{1}{4}\right)\frac{z^5}{5!} - \cdots = \\ &= z - \frac{1}{4}z^3 + \frac{11}{160}z^5 - \cdots; \\ \operatorname{dn} z &= 1 - \frac{1}{2}\frac{z^2}{2!} + \frac{1}{2}\left(4 + \frac{1}{2}\right)\frac{z^4}{4!} - \frac{1}{2}\left(16 + 44 \cdot \frac{1}{2} + \frac{1}{4}\right)\frac{z^6}{6!} + \cdots = \\ &= 1 - \frac{1}{4}z^2 + \frac{3}{32}z^4 - \frac{17}{640}z^6 + \cdots. \end{aligned}$$

Now substitute $\frac{z}{k} = \sqrt{2}z$ for z. Then

$$\begin{aligned} \operatorname{sn}\frac{z}{k} &= \operatorname{sn}(\sqrt{2}z) = \sqrt{2}z - \frac{1}{4}(\sqrt{2}z)^3 + \frac{11}{160}(\sqrt{2}z)^5 - \cdots = \\ &= \sqrt{2}z - \frac{\sqrt{2}}{2}z^3 + \frac{11\sqrt{2}}{40}z^5 - \cdots ; \\ k\operatorname{sn}\frac{z}{k} &= \frac{1}{\sqrt{2}}\operatorname{sn}(\sqrt{2}z) = z - \frac{1}{2}z^3 + \frac{11}{40}z^5 - \cdots = \sum_{n=0}^{\infty} a_n z^n; \\ \operatorname{dn}\frac{z}{k} &= \operatorname{dn}(\sqrt{2}z) = 1 - \frac{1}{4}2z^2 + \frac{3}{32}4z^4 - \frac{17}{640}8z^6 + \cdots = \\ &= 1 - \frac{1}{2}z^2 + \frac{3}{8}z^4 - \frac{17}{80}z^6 + \cdots = \sum_{n=0}^{\infty} b_n z^n . \end{aligned}$$

Power series $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$ converge in the interval $\left|\frac{z}{k}\right| < K$, that is for $|z| < kK = A_3$.

In order to obtain the coefficients c_n in

$$S(z) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < A_3,$$

one uses the standard procedure of dividing series. The result is

$$S(z) = z - \frac{z^5}{10} + \cdots \quad |z| < A_3.$$

The first six terms are the same as those obtained by the successive approximations method.

In order to get the first terms in the series expansion for the function C one uses the formula

$$C(z) = \operatorname{cn}\frac{z}{k}.$$

The first terms of the series expansion for the function $\operatorname{cn} z$ are given in (28). Set $k = \frac{1}{\sqrt{2}}$. Then

$$\begin{aligned} \operatorname{cn} z &= 1 - \frac{z^2}{2!} + \left(1 + 4\cdot\frac{1}{2}\right)\frac{z^4}{4!} - \left(1 + 44\cdot\frac{1}{2} + 16\cdot\frac{1}{4}\right)\frac{z^6}{6!} + \cdots = \\ &= 1 - \frac{1}{2}z^2 + \frac{1}{8}z^4 - \frac{3}{80}z^6 \cdots . \end{aligned}$$

Substitute $\frac{z}{k} = \sqrt{2}z$ for z. Then $\frac{z}{k} = \sqrt{2}z$ and

$$\begin{aligned} C(z) &= \operatorname{cn}(\sqrt{2}z) = 1 - \frac{1}{2}(\sqrt{2}z)^2 + \frac{1}{8}(\sqrt{2}z)^4 - \frac{3}{80}(\sqrt{2}z)^6 \cdots = \\ &= 1 - z^2 + \frac{1}{2}z^4 - \frac{3}{10}z^6 \cdots . \end{aligned}$$

The result is
$$C(z) = 1 - z^2 + \frac{1}{2}z^4 - \frac{3}{10}z^6 \cdots \quad (|z| < A_3).$$

The series expansion for S' can be obtained using the formula
$$S'(z) = C_3(z)\left(1 + S_3^2(z)\right).$$
Making use of the series expansions for $S(z)$ and $C(z)$ one gets
$$S(z)\cdot S(z) = \left(z - \frac{z^5}{10} + \frac{z^9}{120} - \cdots\right)\cdot\left(z - \frac{z^5}{10} + \frac{z^9}{120} - \cdots\right) =$$
$$= z^2 - \left(\frac{1}{10} + \frac{1}{10}\right)z^6 + \left[\frac{1}{120} + \frac{1}{120} + \left(-\frac{1}{10}\right)\cdot\left(-\frac{1}{10}\right)\right]z^{10} - \cdots =$$
$$= z^2 - \frac{1}{5}z^6 + \frac{8}{300}z^{10} - \cdots;$$
$$1 + S^2(z) = 1 + z^2 - \frac{1}{5}z^6 + \frac{8}{300}z^{10} - \cdots;$$

$$S'(z) = C(z)\left(1 + S^2(z)\right) =$$
$$= \left(1 - z^2 + \frac{1}{2}z^4 - \frac{3}{10}z^6 + \cdots\right)\cdot\left(1 + z^2 - \frac{1}{5}z^6 + \frac{8}{300}z^{10} - \cdots\right) =$$
$$= 1 + (1-1)z^2 + \left((-1)\cdot 1 + \frac{1}{2}\cdot 1\right)z^4 + \left(1\cdot\left(-\frac{1}{5}\right) + \frac{1}{2}\cdot 1 + \left(-\frac{3}{10}\right)\cdot 1\right)z^6 + \cdots =$$
$$= 1 - \frac{1}{2}z^4 + \cdots.$$
The first seven terms in the series expansion above are the same as those obtained for the function S' using the method of successive approximations.

Naturally, the more coefficients in the series expansions for the Jacobi elliptic functions are given, the more coefficients in the series expansions for S and related functions can be computed.

Let us consider now the function $F(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}$, which is inverse for $S(t)$. One gets by using the binomial formula that

$$\frac{1}{\sqrt{1-t^4}} = (1-t^4)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-t^4) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1\cdot 2}(-t^4)^2 +$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}(-t^4)^3 +$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{1\cdot 2\cdot 3\cdot 4}(-t^4)^4 + \cdots +$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{1\cdot 2\cdot 3\cdots n}(-t^4)^n + \cdots =$$
$$= 1 + \frac{\frac{1}{2}}{1}t^4 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}t^8 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3}t^{12} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1\cdot 2\cdot 3\cdot 4}t^{16} + \cdots +$$
$$+ \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{2n-1}{2}}{1\cdot 2\cdot 3\cdots n}t^{4n} + \cdots =$$
$$= 1 + \frac{1}{2^1\cdot 1!}t^4 + \frac{1\cdot 3}{2^2\cdot 2!}t^8 + \frac{1\cdot 3\cdot 5}{2^3\cdot 2!}t^{12} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}t^{16} + \cdots +$$
$$+ \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\cdot n!}t^{4n} + \cdots =$$
$$= 1 + \frac{1}{2}t^4 + \frac{1\cdot 3}{2\cdot 4}t^8 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^{12} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}t^{16} + \cdots +$$
$$+ \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 4\cdot 6\cdot 8\cdots 2n}t^{4n} + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}t^{4n},$$

where

$$(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1) \quad (n=1,2,\dots);$$
$$(2n)!! = 2\cdot 4\cdot 6\cdots 2n \quad (n=1,2,\dots).$$

Then

$$F_3(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}} = \int_0^x \left(1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}t^{4n}\right) dt =$$
$$= x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\int_0^x t^{4n}dt = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{x^{4n+1}}{4n+1}.$$

We will show that this series expansion is convergent at the end points $x = \pm 1$ also. For this set $x = 1$ and consider

$$1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{4n+1}. \tag{62}$$

By Rabe's test for series with positive terms the series expansion $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1.$$

In our case

$$r_n = n\left(\frac{(2n-1)!!\,(2n+2)!!\,(4n+5)}{(2n)!!\,(4n+1)\,(2n+1)!!} - 1\right) = \frac{(12n+9)n}{(4n+1)(2n+1)};$$

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\frac{(12n+9)n}{(4n+1)(2n+1)} = \frac{12}{8} = \frac{4}{3} > 1.$$

Hence (62) converges. By symmetry arguments the series expansion formula

$$F(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}} = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{x^{4n+1}}{4n+1}$$

is valid for any $x \in [-1,1]$.

The convergence of the series expansion above is slow as shown by the following observation. We have that

$$A = F(1) = \int_0^1 \frac{dt}{\sqrt{1-t^4}} = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{4n+1} =$$

$$= 1 + \frac{1}{2}\cdot\frac{1}{5} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{13} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{17} + \cdots + \\ + \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 4\cdot 6\cdot 8\cdots 2n}\cdot\frac{1}{4n+1} + \cdots.$$

One obtains performing calculations by Maple8 program the following numerical results:

N	$A_3 \approx 1 + \sum\limits_{n=1}^{N}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{4n+1}$
10	1,225016205
50	1,271430960
100	1,282924587
200	1,291118976
300	1,294762353
400	1,296937245
700	1,300372304
1000	1,302111500

At the end we put it down the recurrent formula for higher order derivatives for the function $S(t)$, which is obtained using the Leibnitz formula for derivatives as given in [10][Ch. 3, § 6]:

$$\begin{aligned} S^{(n)} &= -2(S^3)^{(n-2)} = -2(S^2\cdot S)^{(n-2)} = -2\textstyle\sum_{k\le n-2}(S^2)^{(k)}\cdot S^{(n-2-k)} \\ &= -2\textstyle\sum_{k\le n-2}\binom{n-2}{k}\left(\sum_{i\le k}\binom{k}{i}S^{(i)}\cdot S^{(k-i)}\right)\cdot S^{(n-2-k)}. \end{aligned}$$

It can be used to calculate terms in the Taylor series expansion for $S(t)$.

9 APPENDIX

We collect here various formulae and relations for the functions $S(t)$, $C(t)$. Proofs of the relations which were not proved earlier are given below.

9.1 Periodicity

$$S(t) = S(t+4A), \quad C(t) = C(t+4A)$$

9.2 Sign properties

$$S(-t) = -S(t), \quad C(-t) = C(t)$$

9.3 Reduction formulae

$$\begin{array}{ll} S(A+t) = C(t), & C(A+t) = -S(t), \\ S(2A+t) = -S(t), & C(2A+t) = -C(t), \\ S(3A+t) = -C(t), & C(3A+t) = S(t). \end{array} \tag{63}$$

9.4 Sum formulae

$$S(\alpha+\beta) = \frac{S(\alpha)C(\beta) + S(\beta)C(\alpha)}{1 - S(\alpha)S(\beta)C(\alpha)C(\beta)} \tag{64}$$

$$C(\alpha+\beta) = \frac{C(\beta)C(\alpha) - S(\alpha)S(\beta)}{1 + S(\alpha)S(\beta)C(\alpha)C(\beta)} \tag{65}$$

We mention also

$$S(\alpha) + S(\beta) = \frac{2S\left(\frac{\alpha+\beta}{2}\right)C\left(\frac{\alpha-\beta}{2}\right) + 2S\left(\frac{\alpha-\beta}{2}\right)C\left(\frac{\alpha+\beta}{2}\right)\cdot Q}{1-Q^2},$$

$$S(\alpha) - S(\beta) = \frac{2S\left(\frac{\alpha-\beta}{2}\right)C\left(\frac{\alpha+\beta}{2}\right) + 2S\left(\frac{\alpha+\beta}{2}\right)C\left(\frac{\alpha-\beta}{2}\right)\cdot Q}{1-Q^2},$$

$$C(\alpha) + C(\beta) = \frac{2C\left(\frac{\alpha+\beta}{2}\right)C\left(\frac{\alpha-\beta}{2}\right) + 2S\left(\frac{\alpha+\beta}{2}\right)S\left(\frac{\alpha-\beta}{2}\right)\cdot Q}{1-Q^2},$$

$$C(\alpha) - C(\beta) = -\frac{2S\left(\frac{\alpha+\beta}{2}\right)S\left(\frac{\alpha-\beta}{2}\right) + 2C\left(\frac{\alpha+\beta}{2}\right)C\left(\frac{\alpha-\beta}{2}\right)\cdot Q}{1-Q^2}.$$

where $Q := S\left(\frac{\alpha+\beta}{2}\right)S\left(\frac{\alpha-\beta}{2}\right)C\left(\frac{\alpha+\beta}{2}\right)C\left(\frac{\alpha-\beta}{2}\right)$.

Proof. Let us prove the first of the above four relations. Proof for the remaining three can be conducted in a similar manner.

One has using (64) and the sign properties of $S(t)$ and $C(t)$ that

$$S(x+y) = \frac{S(x)C(y)+S(y)C(x)}{1-S(x)S(y)C(x)C(y)}$$

and

$$S(x-y) = \frac{S(x)C(y)-S(y)C(x)}{1+S(x)S(y)C(x)C(y)}.$$

Introduce

$$\alpha := x+y, \quad \beta := x-y,$$
$$x = \tfrac{\alpha+\beta}{2}, \quad y = \tfrac{\alpha-\beta}{2}.$$

Then

$$S(\alpha)+S(\beta) = \frac{S(x)C(y)+S(y)C(x)}{1-S(x)S(y)C(x)C(y)} + \frac{S(x)C(y)-S(y)C(x)}{1+S(x)S(y)C(x)C(y)} =$$
$$= \frac{1}{1-Q^2}[S(x)C(y)+S(y)C(x)+QS(x)C(y)+QS(y)C(x)+$$
$$+S(x)C(y)-S(y)C(x)-QS(x)C(y)+QS(y)C(x)] =$$
$$= \frac{2S(x)C(y)+2S(y)C(x)\cdot Q}{1-Q^2}.$$

9.5 Double argument formulae

$$S(2\alpha) = \frac{2S(\alpha)C(\beta)}{1-S^2(\alpha)C^2(\alpha)} \tag{66}$$

$$C(2\alpha) = \frac{C^2(\alpha)-S^2(\alpha)}{1+S^2(\alpha)C^2(\alpha)}. \tag{67}$$

9.6 Multiplication formulae

Set $u := S(\alpha)\cdot C(\beta)$ and $v := S(\beta)\cdot C(\alpha)$. Then the following relations are valid:

$$\begin{aligned} 2u &= S(\alpha+\beta)+S(\alpha-\beta)+uv\cdot[S(\alpha-\beta)-S(\alpha+\beta)] \\ 2v &= S(\alpha+\beta)-S(\alpha-\beta)-uv\cdot[S(\alpha+\beta)+S(\alpha-\beta)] \end{aligned} \tag{68}$$

Proof. By combining the formula (64) and

$$S(\alpha-\beta) = \frac{S(\alpha)C(\beta)-S(\beta)C(\alpha)}{1+S(\alpha)S(\beta)C(\alpha)C(\beta)}.$$

Set $p := S(\alpha)\cdot S(\beta)$ and $q := C(\alpha)\cdot C(\beta)$. Then the following relations are valid:

$$\begin{aligned} 2p &= C(\alpha-\beta)-C(\alpha+\beta)-pq\cdot[C(\alpha+\beta)+C(\alpha-\beta)] \\ 2q &= C(\alpha+\beta)+C(\alpha-\beta)+pq\cdot[C(\alpha+\beta)-C(\alpha-\beta)] \end{aligned} \tag{69}$$

Proof. By combining the formula (65) and

$$C(\alpha-\beta) = \frac{C(\beta)C(\alpha)+S(\alpha)S(\beta)}{1-+S(\alpha)S(\beta)C(\alpha)C(\beta)}.$$

9.7 Unit circle formula

$$S^2(t) + S^2(t)C^2(t) + C^2(t) = 1 \text{ for any } t \in R \tag{70}$$

9.8 Formulae involving derivatives

$$S' = C(1 + S^2) \tag{71}$$

$$C' = -S(1 + C^2) \tag{72}$$

$$(SC)' = C^2 - S^2 \tag{73}$$

Proof. By computation using (71) and (72).

$$\left(\frac{S}{C}\right)' = \frac{1 + S^2C^2}{C^2} \tag{74}$$

Proof. By computation using (71) and (72).

$$\left(\frac{C}{S}\right)' = -\frac{1 + S^2C^2}{S^2} \tag{75}$$

Proof. By computation using (71) and (72).

9.9 Special values

We present special values of the functions $S(t)$ and $C(t)$ in the interval $(0, A)$. Special values outside can be obtained using the reduction formulae (63) and sign/periodicity properties of the functions $S(t)$ and $C(t)$.

$$\begin{array}{lll} S(A/3) = \frac{1+\sqrt{3}-\sqrt{2}\,\sqrt[4]{3}}{2}, & S(A/2) = \sqrt{\sqrt{2}-1}, & S(2A/3) = \kappa S(A/3), \\ C(A/3) = \kappa C(2A/3), & C(A/2) = \sqrt{\sqrt{2}-1}, & C(2A/3) = \frac{1+\sqrt{3}-\sqrt{2}\,\sqrt[4]{3}}{2}, \end{array} \tag{76}$$

where $\kappa = \sqrt{\sqrt{3} + \sqrt{2}\,\sqrt[4]{3}}$. The approximate values are

$$\frac{1 + \sqrt{3} - \sqrt{2}\,\sqrt[4]{3}}{2} \approx 0,43542054, \quad \kappa \approx 1,89558975.$$

Proof. Set $\alpha := A/3$. Denote $u := S(2\alpha)$, $w := S(\alpha)$, $v := C(2\alpha)$, $z := C(\alpha)$, $\Pi = uwvz$. One obtains using the sum formulae (64) and (65) and sign properties of $S(t)$ and

$C(t)$ that

$$\begin{aligned} 1 &= S(2\alpha+\alpha) = \frac{uz+wv}{1-\Pi} \\ S(\alpha) &= S(2\alpha-\alpha) = \frac{uz-wv}{1+\Pi} \\ 0 &= C(2\alpha+\alpha) = \frac{vz-uw}{1+\Pi} \\ C(\alpha) &= C(2\alpha-\alpha) = \frac{vz+uw}{1-\Pi}. \end{aligned} \tag{77}$$

Then (77) writes as

$$1 = \frac{uz+wv}{1-\Pi}, \quad w = \frac{uz-wv}{1+\Pi}, \quad vz = uw, \quad z = \frac{vz+uw}{1-\Pi} \tag{78}$$

or

$$\begin{aligned} &1 - uwvz = uz + wv, \\ &1 + uwvz = \frac{uz-wv}{w}, \\ &\frac{u}{v} = \frac{z}{w}, \\ &1 - uwvz = \frac{vz+uw}{z}. \end{aligned} \tag{79}$$

Introduce $\kappa > 0$ by $u = \kappa v$, $z = \kappa w$ and rewrite (77) as

$$\begin{aligned} &1 - \kappa^2 v^2 w^2 = (1+\kappa^2) vw, \\ &1 + \kappa^2 v^2 w^2 = (\kappa^2 - 1) v, \\ &1 - \kappa^2 v^2 w^2 = 2v. \end{aligned} \tag{80}$$

Solving the system above gives

$$v = w = \frac{1+\sqrt{3}-\sqrt{2}\,\sqrt[4]{3}}{2}, \quad \kappa = \sqrt{\sqrt{3}+\sqrt{2}\,\sqrt[4]{3}}.$$

Notice that $v = w$ is one of the roots of the equation

$$Z^4 - 2Z^3 - 2Z + 1 = 0.$$

To find $S(A/2)$ and $C(A/2)$ notice that

$$0 = C(A) = \frac{C^2\left(\frac{A}{2}\right) - S^2\left(\frac{A}{2}\right)}{1 - S^2\left(\frac{A}{2}\right)C^2\left(\frac{A}{2}\right)}, \quad 1 = S(A) = \frac{2S\left(\frac{A}{2}\right)C\left(\frac{A}{2}\right)}{1 - S^2\left(\frac{A}{2}\right)C^2\left(\frac{A}{2}\right)} \tag{81}$$

((66) and (67)) and hence $C\left(\frac{A}{2}\right) = S\left(\frac{A}{2}\right)$ (both $C\left(\frac{A}{2}\right) > 0$ and $S\left(\frac{A}{2}\right) > 0$). One obtains from the second equation in (81) that

$$1 - S^4\left(\frac{A}{2}\right) = 2S^2\left(\frac{A}{2}\right),$$

and finally

$$S\left(\frac{A}{2}\right) = \sqrt{\sqrt{2}-1} = C\left(\frac{A}{2}\right).$$

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А. Грицан, Ф.Ж. Садырбаев. О функциях, возникающих в теории уравнения Эмдена – Фаулера.

Аннотация. Рассматриваются функции, которые являются решениями уравнения $x'' = -2x^3$ и им подобные. Решения, удовлетворяющие начальным условиям $x(0) = 0$, $x'(0) = 1$ и $x(0) = 1$, $x'(0) = 0$ известны как соответственно лемнискатные синус и косинус. По аналогии с теорией элементарных тригонометрических функций, доказываются наиболее примечательные свойства и приводятся разнообразные формулы, связывающие лемнискатные функции и из производные. Всюду, где возможно, подчеркивается аналогия с тригонометрическими функциями. Основные технические средства - теория эллиптических функций Якоби.

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A. Gricans, F.Sadirbajevs. Par funkcijam kuras rodas Emdena - Fowlera diferenciālvienādojuma teorijā.

Anotācija. Apskatītas sinusam un kosinusam lidzīgas funkcijas, kurās apmierina nelinearu DV $x'' = -2x^3$.

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