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CHARACTERISTIC NUMBERS OF NON-AUTONOMOUS EMDEN-FOWLER TYPE EQUATIONS

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Abstract. We consider the Emden–Fowler equation $x'' = -q(t)|x|^{2\varepsilon}x$, $\varepsilon > 0$, in the interval $[a, b]$. The coefficient $q(t)$ is a positive valued continuous function. The Nehari characteristic number λ_n associated with the Emden–Fowler equation coincides with a minimal value of the functional $\frac{\varepsilon}{1+\varepsilon}\int_a^b x'^2(t)\,dt$ over all solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has exactly } (n-1) \text{ zeros in } (a, b).$$

The respective solution is called the Nehari solution. We construct an example which shows that the Nehari extremal problem may have more than one solution.

Key words: Characteristic numbers, Emden–Fowler equation, Nehari's solutions

1. Nehari's Solutions

Behavior of solutions of the Emden–Fowler type equation

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0, \tag{1.1}$$

where $q(t)$ is a positive valued continuous function, may be complicated if $q(t)$ is a non-monotone function.

Some regularity to the theory of the Emden–Fowler type equations of the form (1.1) is brought by the so called Nehari's solutions. The Nehari theory applies to equations of the type (1.1). The general theorem by Nehari ([2, Theorem 3.2]) when adapted to the case under consideration states that the extremal problem.

$$H(x) = \int_a^b \left[x'^2 - (1+\varepsilon)^{-1} q(t) x^{2+2\varepsilon}\right] dt \to \inf, \quad x \in \Gamma_n, \tag{1.2}$$

has a solution. Here Γ_n consists of all functions $x(t)$, which are continuous and piece-wise continuously differentiable in $[a, b]$, also there exist numbers a_ν such that

$$a = a_0 < a_1 < \ldots < a_n = b;$$

for $\nu = 0, \ldots, n$, $x(a_\nu) = 0$, but $x \not\equiv 0$ in any $[a_{\nu-1}, a_\nu]$, and

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\, dt = \int_{a_{\nu-1}}^{a_\nu} q(t) x^2 |x|^{2\varepsilon}\, dt. \tag{1.3}$$

A respective extremal function $x_n(t)$ must be twice continuously differentiable solution of equation (1.1), which vanishes at the points $t = a$ and $t = b$ and has exactly $(n-1)$ zeros in (a, b). Any such solution of equation (1.1) satisfies condition (1.3).

By combining (1.3) with (1.2) one gets that

$$\lambda_n(a,b) := \min_{x \in \Gamma_n} H(x) = H(x_n) = \frac{\varepsilon}{1+\varepsilon} \int_a^b q(t) x_n^{2+2\varepsilon}\, dt = \frac{\varepsilon}{1+\varepsilon} \int_a^b x_n'^2(t)\, dt.$$

Thus the characteristic number $\lambda_n(a,b)$ is (up to a multiplicative constant) a minimal value of the functional $\int_a^b x'^2(t)\, dt$ over the set of all solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon} x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has } (n-1) \text{ zeros in } (a,b).$$

The characteristic numbers λ_n are called *the Nehari numbers* and the respective solutions of the differential equation are called *the Nehari solutions.*

Remark 1. Nehari's numbers $\lambda_n(a,b)$ are uniquely defined by the interval (a, b). In [2] Nehari mentioned that the theory could be developed much easier if the associated Nehari's solution be unique. It was shown theoretically in [3] that this is not the case. There exist equations of the type (1.1), which have more than one Nehari's solution for certain parameters a, b and n.

2. Nonuniqueness of the Nehari's Solutions

In this section we construct the Emden–Fowler equation of the form (2.2) which possesses three solutions which obey conditions (2.3). Two of them are Nehari's solutions.

2.1. Lemniscatic functions

We use in our considerations the so called lemniscatic functions which can be defined as solutions of the equation

$$x'' = -2x^3. \tag{2.1}$$

The functions $\operatorname{sl} t$ and $\operatorname{cl} t$ solve equation (2.1) and satisfy respectively the following initial conditions

$$x(0) = 0, \quad x'(0) = 1 \quad \text{and} \quad x(0) = 1, \quad x'(0) = 0.$$

The lemniscatic sine and cosine functions are periodic with a minimal period $4A$, where $A = \int\limits_0^1 \frac{ds}{\sqrt{1-s^4}} \approx 1.311$. For convenience of reference we mention the following properties of these functions:

$$\operatorname{sl} 0 = \operatorname{sl} 2A = 0, \quad \operatorname{sl} A = 1,$$
$$\operatorname{cl} 0 = 1,\ \operatorname{cl} A = 0,\ \operatorname{cl} 2A = -1,$$
$$\operatorname{sl}' t = \operatorname{cl} t\,(1 + \operatorname{sl}^2 t), \quad \operatorname{cl}' t = -\operatorname{sl} t(1 + \operatorname{cl}^2 t), \quad \lim_{t\to 0} \frac{\operatorname{sl} t}{t} = 1.$$

More properties and useful formula of these functions are given in [1].

Equation. Consider the boundary value problem

$$x'' = -q(t)\,x^3, \tag{2.2}$$

$$x(-1) = 0, \quad x(1) = 0, \quad x(t) > 0,\ t \in (-1, 1). \tag{2.3}$$

Let $q(t) = 2/\xi^6(t)$, where

$$\xi(t) = \begin{cases} \xi_1(t) = ht + \eta, & -1 \le t \le 0, \\ \xi_2(t) = -ht + \eta, & 0 \le t \le 1. \end{cases}$$

Thus $\xi(t)$ is a "Λ-shaped" piece-wise linear function, which depends on a positive valued parameter h, $\eta := h + 1$.

Solutions. A solution (solutions) of problem (2.2)–(2.3) can be composed of solutions of the following two problems

$$x_1'' = -\frac{k}{(ht+\eta)^6}\,x_1^3, \quad x_1(-1) = 0,\ x_1(0) = \tau, \quad x_1(t) > 0,\ t \in (-1, 0), \tag{2.4}$$

$$x_2'' = -\frac{k}{(-ht+\eta)^6}\,x_2^3, \quad x_2(0) = \tau,\ x_2(1) = 0, \quad x_2(t) > 0,\ t \in (0, 1),$$

where $\tau > 0$. The function

$$x(t) = \begin{cases} x_1(t), & \text{if } -1 \le t \le 0, \\ x_2(t), & \text{if } 0 \le t \le 1 \end{cases}$$

is a C^2-solution of problem (2.2)–(2.3) if additionally the smoothness condition

$$x_1'(0) = x_2'(0)$$

is satisfied. Problem (2.4) has a solution

$$x_1(t, \beta_1) = \beta_1^{\frac{1}{2}}(ht + \eta)\ \mathrm{sl}\left(\beta_1^{\frac{1}{2}}\frac{t+1}{ht+\eta}\right),$$

where $\beta_1 = x_1'(-1) > 0$ is such that $x_1(0; \beta_1) = \tau$. The derivative of x_1 is given by

$$x_1'(t; \beta_1) = \beta_1^{\frac{1}{2}} h\ \mathrm{sl}\left(\beta_1^{\frac{1}{2}}\frac{t+1}{ht+\eta}\right) + \beta_1 \frac{-h+\eta}{ht+\eta}\ \mathrm{sl}'\left(\beta_1^{\frac{1}{2}}\frac{t+1}{ht+\eta}\right).$$

Similar formulas are valid for $x_2(t)$. Notice that

$$x_2'(1) = -\beta_2 < 0.$$

In order to get explicit formula for a solution of the BVP (2.2)–(2.3) one has to solve a system of two equations with respect to (β_1, β_2)

$$\begin{cases} x_1(0; \beta_1) = x_2(0; \beta_2), \\ x_1'(0; \beta_1) = x_2'(0; \beta_2). \end{cases}$$

This system after replacements and simplifications looks as

$$\begin{cases} \beta_1^{\frac{1}{2}}\ \mathrm{sl}\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) = \beta_2^{\frac{1}{2}}\ \mathrm{sl}\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right), \\ \beta_1^{\frac{1}{2}} h\ \mathrm{sl}\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) + \dfrac{\beta_1}{\eta}\ \mathrm{sl}'\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) = -\beta_2^{\frac{1}{2}} h\ \mathrm{sl}\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right) - \dfrac{\beta_2}{\eta}\ \mathrm{sl}'\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right), \end{cases} \tag{2.5}$$

where $0 < \dfrac{\beta_1^{\frac{1}{2}}}{\eta}, \dfrac{\beta_2^{\frac{1}{2}}}{\eta} < 2A$. In new variables $u := \dfrac{\beta_1^{\frac{1}{2}}}{\eta}$, $v := \dfrac{\beta_2^{\frac{1}{2}}}{\eta}$ the system takes the form

$$\begin{cases} \Phi(u) = \Phi(v), & 0 < u,\ v < 2A, \\ \Psi_h(u) = -\Psi_h(v), & h > 0, \end{cases} \tag{2.6}$$

where

$$\Phi(z) := z\ \mathrm{sl}\, z, \quad \Psi_h(z) := hz\ \mathrm{sl}\, z + z^2\, \mathrm{sl}'\, z.$$

Notice that if a solution $(\bar{u}, \bar{v})$ of system (2.6) exists, then a solution $x(t)$ of the BVP (2.2)–(2.3) can be constructed such that

$$x'(-1) = \beta_1 = \bar{u}^2(h+1)^2, \quad x'(1) = -\beta_2 = -\bar{v}^2(h+1)^2.$$

Proposition 1. *For h large enough system (2.6) has exactly three solutions:*

1. *There exists a unique solution of the form (u_0, u_0). One has that*

$$(u_0, u_0) \to (2A, 2A), \quad h \to +\infty.$$

2. *In the triangle $\{0 < u, v < 2A,\ v > u)\}$ there exists a unique solution (u_1, v_1) such that*

$$(u_1, v_1) \to (0, 2A) \quad h \to +\infty.$$

3. *In the triangle $\{0 < u, v < 2A,\ v < u)\}$ there exists a unique solution (u_2, v_2). Solutions (u_1, v_1) and (u_2, v_2) are symmetric, that is, $(v_2, u_2) = (u_1, v_1)$.*

2.2. Investigation of a system

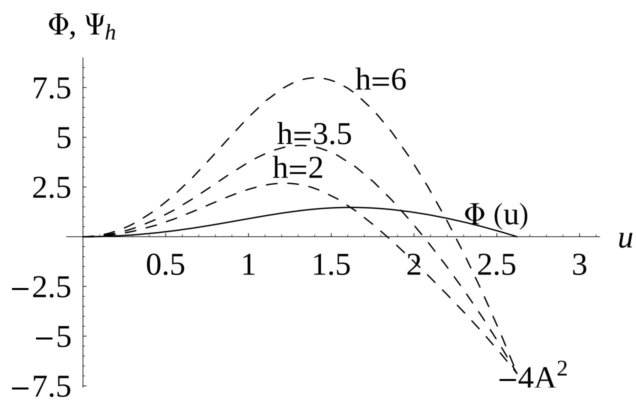

Figure 1. Functions $\Phi(u)$ (solid line) and $\Psi_h(u)$ (dashed lines).

Standard analysis shows that function $\Phi(z) = z \operatorname{sl} z$ has the following properties (see Fig. 1):

$$\Phi(0) = \Phi(2A) = 0, \quad \Phi(z) > 0,\ \forall z \in (0, 2A),$$

$$\Phi'(z) = \operatorname{sl} z + z \operatorname{sl}' z,$$

$\Phi_{max} = \Phi(z_0) \approx 1.47233$ at the unique point of maximum $z_0 \approx 1.61879$.

Consider a set of zeros of a function $\Phi(u) - \Phi(v)$ in the square $Q = \{(u, v) : 0 \le u, v \le 2A\}$. It consists of the diagonal segment Γ_0 $(u = v)$ and two symmetric branches Γ_+ and Γ_-, which are shown in Fig. 2.

Lemma 1. *The relation $F(u, v) = \Phi(u) - \Phi(v) = 0$ defines a function $v = f(u)$ for $u \in [0, 2A]$. One has that*

$$f(0) = 2A, \quad f(2A) = 0, \quad f'_u = \frac{\Phi(u)}{\Phi(v)}|_{v=f(u)} < 0, \quad u \in [0, 2A].$$

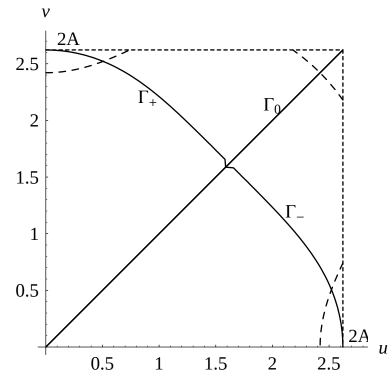

Figure 2. Zeros of $\Phi(u)-\Phi(v))$ (solid line) and $\Psi_h(u)+\Psi_h(v)$ (dashed line), $h=12$.

Proof. A set of zeros of $F(u,v)$ for $v>u$ (branch Γ_+ in Fig. 2) can be parameterized by the equalities $\Phi(u)=p,\ \Phi(v)=p$, where $p\in[0,\Phi_{max}]$. If p changes from 0 to Φ_{max}, variables u and v respectively increase from 0 to z_0 and decrease from $2A$ to z_0. One gets by using Implicit Function Theorem that there exists function $v=f(u),\ u\in(0,z_0)$ such that $F(u,f(u))=0$ for $u\in(0,z_0)$ and

$$f'_u(u) = -\left.\frac{\frac{\partial F}{\partial u}(u;v)}{\frac{\partial F}{\partial v}(u;v)}\right|_{v=f(u)} = \left.\frac{\Phi'(u)}{\Phi'(v)}\right|_{v=f(u)},$$

Since $\Phi'(u)>0$ for $u\in(0,z_0)$ and $\Phi'(v)<0$ for $v\in(z_0,2A)$ one has that $f'_u(u)<0$. The graph of $f(u)$ is the set Γ_+.

The same type of arguments can be applied for (u,v) in the lower triangle, $u>v$. Thus a decreasing function $v=f(u)$ exists for $u\in[0,2A]$. The graph of this function is the union of Γ_+ and Γ_-. ■

Consider function $\Psi_h(z)$. We mention the following properties:

$$\Psi_h(0)=0, \quad \Psi_h(2A)=-4A^2 \quad \forall h>0,$$
$$\Psi'_h(z)=h\Phi'(z)+z\Phi''(z).$$

Function $\Psi_h(z)$ increases for $z\in(0,z_{max}(h))$ and decreases for $z\in(z_{max},2A)$. It is easy to show that

$$(\Psi_h)_{max}=\Psi_h(z_{max})\to+\infty, \quad z_{max}(h)\to z_0, \quad h\to+\infty.$$

Lemma 2. *For large h a set Z of zeros of the function $\Psi_h(u)+\Psi_h(v)$ in the square Q consists of three mutually disjointed sets*

$$Z_+\subset\{(x,y):\ 0\le x\le\delta,\ 2A-\delta\le y\le 2A\},$$
$$Z_0\subset\{(x,y):\ 2A-\delta\le x\le 2A,\ 2A-\delta\le y\le 2A\},$$
$$Z_-\subset\{(x,y):\ 2A-\delta\le x\le 2A,\ 0\le y\le\delta\}.$$

It is also true that $\delta \to 0$ as $h \to +\infty$.

Proof. Let z_1 be a unique zero of $\Psi_h(z)$ in the interval $(0, 2A)$. Let z_* and z^* be the level points defined by the relations

$$\Psi_h(z_*) = \Psi_h(z^*) = 4A^2, \quad z_* < z^*.$$

It is clear that the equality $\Psi_h(x) + \Psi_h(y) = 0$ implies the inclusion $(x, y) \in (0, z_*) \bigcup (z^*, 2A)$. Indeed, if $x \in (z_*, z^*)$ then $\Psi_h(x) > 4A^2$ and $\Psi_h(x) + \Psi_h(y) > 0$ for any y. If $\Psi_h(x) + \Psi_h(y) = 0$ then either x or y belongs to $(z_1, 2A)$.

Consider the case $x \in (z_1, 2A)$. Then there are two values of y, say, y_1 and y_2, such that

$$\Psi_h(x) + \Psi_h(y) = 0, \quad y_1 \in (0, z_*), \quad y_2 \in (z*, z_1).$$

Similarly, if $y \in (z_1, 2A)$, then there are two values of x, x_1 and x_2, such that $x_1 \in (0, z_*)$ and $x_2 \in (z^*, z_1)$. Therefore any point $(x, y) \in Q$ such that $\Psi_h(x) + \Psi_h(y) = 0$ belongs to one of the sets Z_+, Z_0 or Z_-.

Let us show that $z_* \to 0$ and $z^* \to 2A$ as $h \to +\infty$. Both values of z satisfy the relation $\Psi_h(z) = hz \operatorname{sl} z + z^2 \operatorname{sl}' z = 4A^2$. Then $z \operatorname{sl} z + \frac{1}{h} z^2 \operatorname{sl}' z = \frac{1}{h} 4A^2$. If $h \to +\infty$ then $z \operatorname{sl} z \to 0$ and the level points $z_*(h)$ and $z^*(h)$ tend respectively to 0 and $2A$. ■

Lemma 3. *The relation $G_h(u, v) = \Psi_h(u) + \Psi_h(v) = 0$ defines a function $v = g(u)$ for $u \in [0, \delta]$. One has that $g'_u = -\frac{\Psi'_h(u)}{\Psi'_h(v)}|_{v=g(u)} > 0$ for $u \in [0, \delta]$.*

Proof. (of Proposition 2.1) Consider the set Z_+. The function $v = f(u)$ strictly decreases and satisfies the relation $f(0) = 2A$. The function $v = g(u)$ strictly increases and satisfies the relations $g(0) < 2A$, $g(u_*) = 2A$ for some $u_* \in (0, \delta)$. Therefore there exists a unique point of intersection of the graphs of both functions in Z_+. By symmetry with respect to the diagonal, the same is true for the set Z_-. Thus two solutions of the system (2.6). For $u = v$ the system (2.6) reduces to a single equation $\Psi_h(z) = 0$, which has a unique solution, tending to $2A$ as $h \to +\infty$. Thus exactly three solutions of the system (2.6).

We also give an alternative proof. Let us parameterize the upper left branch Γ_+ (for this branch $v > u$) by $\Phi(u) = \Phi(v) = p$, where $0 < p < p_*$, $p_* = \max_{[0,2A]}\{\Phi(u)\}$. Function $\Phi(u)$ attains its maximal value $p_* \approx 1.47233$ at the point $m_* \approx 1.61879$. This branch is then defined parametrically as $u = u(p)$, $v = v(p)$. Notice that $(u(0), v(0)) = (0, 2A)$ and $(u(p_*), v(p_*)) = (m_*, m_*)$.

Suppose that $h > 1$ and consider the one argument function

$$\omega(p) := hu(p) \operatorname{sl} u(p) + u^2(p) \operatorname{sl}' u(p) + hv(p) \operatorname{sl} v(p) + v^2(p) \operatorname{sl}' v(p)$$

in the interval $[0, p_*]$. Our intent is to show that this function changes sign only once. Then there exists a unique solution of the system (2.6) on Γ_+ and as a consequence, there exist exactly three solutions of the system (2.6) for $0 < u, v < 2A$.

Since $u(p)\,\mathrm{sl}\,u(p) = v(p)\,\mathrm{sl}\,v(p) = p$, the function ω takes the form

$$\omega(p) = 2hp + u^2(p)\ \mathrm{sl}'\,u(p) + v^2(p)\ \mathrm{sl}'\,v(p).$$

The problem is to show that function $\omega(p)$ strictly increases on the interval $(0, p_*)$, where $u(p)$ is defined parametrically as $u\ \mathrm{sl}\,u = p, \quad u \in (0, m_*)$, and $v(p)$ is defined by $v\ \mathrm{sl}\,v = p, \quad v \in (m_*, 2A)$. Consider the first equation in (2.6). Define two functions $x(p)$ and $y(p)$ parametrically using the equalities

$$x\,\mathrm{sl}\,x = y\,\mathrm{sl}\,y = p \tag{2.7}$$

where $p \in [0, p_*]$, $x : [0, p_*] \to [0, m_*]$, $y : [0, p_*] \to [m_*, 2A]$. The functions $x(p)$ and $y(p)$ are well defined, continuous, but may have infinite derivatives. One has from (2.7) that

$$\frac{dx}{dp} = \frac{1}{\mathrm{sl}\,x + x\,\mathrm{sl}'\,x}, \qquad \frac{dy}{dp} = \frac{1}{\mathrm{sl}\,y + x\,\mathrm{sl}'\,y}.$$

Thus $x(p)$ has infinite derivatives at $p = 0$ and $p = p_*$, and $y(p)$ has infinite derivative at $p = p_*$.

Consider now the second equation in (2.6). We will show that the function

$$\begin{aligned}
\omega(p) &= h\,x\,\mathrm{sl}\,x + x^2\,\mathrm{sl}'\,x + h\,y\,\mathrm{sl}\,y + y^2\,\mathrm{sl}'\,y \\
&= h\,(x\,\mathrm{sl}\,x + y\,\mathrm{sl}\,y) + x^2\,\mathrm{sl}'\,x + +y^2\,\mathrm{sl}'\,y \\
&= 2hp + x^2\,\mathrm{sl}'\,x + +y^2\,\mathrm{sl}'\,y
\end{aligned}$$

is strictly increasing in p for h large enough. One has that

$$\begin{aligned}
\frac{d\,\omega}{dp}(p) &= 2h + 2xx'\ \mathrm{sl}'\,x + x^2\,\mathrm{sl}''\,x\,x' + 2yy'\ \mathrm{sl}'\,y + y^2\,\mathrm{sl}''\,y\,y' \\
&= 2h + xx'(2\,\mathrm{sl}'\,x + x\,\mathrm{sl}''\,x) + yy'(2\,\mathrm{sl}'\,y + y\,\mathrm{sl}''\,y).
\end{aligned} \tag{2.8}$$

Since $\mathrm{sl}\,x$ is a bounded periodic function together with the derivatives $\mathrm{sl}'\,x$ and $\mathrm{sl}''\,x$, the expressions in parentheses are bounded.

Let us evaluate the products $x(p)x'(p)$ and $y(p)y'(p)$. One has for the second one that the value of derivative $\dfrac{dy}{dp} = \dfrac{1}{\mathrm{sl}\,y + x\,\mathrm{sl}'\,y}$ at $p = 0$ is given by

$$\frac{dy}{dp} = \frac{1}{\mathrm{sl}\,2A + 2A\,\mathrm{sl}'\,2A} = -\frac{1}{2A}, \quad y(0)\frac{dy}{dp}(0) = 2A \cdot (-\frac{1}{2A}) = -1.$$

Using the l'Hospital's rule to evaluate the limit yields

$$\begin{aligned}
\lim_{p\to 0} x(p)x'(p) &= \lim_{p\to 0} \frac{x'(p)}{\frac{1}{x(p)}} = \lim_{p\to 0} \frac{x''(p)}{\frac{-1}{x^2(p)}x'(p)} \\
&= \lim_{p\to 0} \frac{2\,\mathrm{sl}'\,x + x\,\mathrm{sl}''\,x}{(\mathrm{sl}^2\,x + 2\,\mathrm{sl}\,x \cdot \mathrm{sl}'\,x \cdot x + x^2\,\mathrm{sl}'\,x) \cdot \frac{1}{x^2}} \\
&= \lim_{p\to 0} \frac{2\,\mathrm{sl}'\,x - 2x\,\mathrm{sl}^3\,x}{(\frac{\mathrm{sl}\,x}{x})^2 + 2(\frac{\mathrm{sl}\,x}{x})\,\mathrm{sl}'\,x + \mathrm{sl}'\,x} = \frac{2\,\mathrm{sl}'\,0}{1 + 2\,\mathrm{sl}'\,0 + \mathrm{sl}'\,0} = \frac{1}{2}
\end{aligned}$$

Similarly can be shown that $\lim_{p\to p_*} x(p)x'(p)$ and $\lim_{p\to p_*} y(p)y'(p)$ are finite. Then the last two addends in (2.8) are finite in the interval $[0, p_*]$ and for h large enough $\dfrac{d\,\omega(p)}{dp}$ is positive. Since $\omega(0) < 0$ and $\omega(p_*) > 0$, this function can change sign only once. Thus only one zero of the system (2.6) in the upper triangle. Totally we have exactly three solutions. ■

2.3. Integrals

The Nehari number $\lambda(-1,1)$ we are looking for is the minimal value of the functional

$$H(x) = \frac{1}{2}\int_{-1}^{1} x'^2(t)\,dt = \frac{1}{2}\int_{-1}^{1} q(t)x^4(t)\,dt$$

over all solutions of the BVP. Notice that

$$H(x) = \frac{1}{2}\int_{-1}^{0} q_1(t)x_1^4(t)dt + \frac{1}{2}\int_{0}^{1} q_2(t)x_2^4(t)dt = J_1 + J_2, \quad q_i(t) = \frac{1}{\xi_i^6(t)}, \ i = 1, 2.$$

Computation yields

$$H(x) = \frac{\beta_1^{\frac{3}{2}}}{3}\left[\frac{\beta_1^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right)\right] + \frac{\beta_2^{\frac{3}{2}}}{3}\left[\frac{\beta_2^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\right],$$

where β_1 and β_2 solves the system (2.5), $\eta = h + 1$.

If $x(t)$ is a "symmetric" solution, then $\beta_1 = \beta_2 = \beta_0$, and the above formula looks as

$$H(x) = 2\,\beta_0^{\frac{3}{2}}\int_0^{\beta_0^{\frac{1}{2}}/\eta} \mathrm{sl}^4(z)\,dz = \frac{2}{3}\,\beta_0^{\frac{3}{2}}\left[\frac{\beta_0^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\right].$$

One has that

$$\frac{\beta_1^{\frac{1}{2}}}{\eta} \xrightarrow[h\to+\infty]{} 0, \qquad \frac{\beta_0^{\frac{1}{2}}}{\eta} \xrightarrow[h\to+\infty]{} 2A, \qquad \frac{\beta_2^{\frac{1}{2}}}{\eta} \xrightarrow[h\to+\infty]{} 2A.$$

$$\lim_{h\to+\infty}\frac{H_{sym}}{H_{asym}} =$$

$$\lim_{h\to+\infty}\frac{\frac{2}{3}\,\beta_0^{\frac{3}{2}}\left[\frac{\beta_0^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\right]}{\frac{1}{3}\,\beta_1^{\frac{3}{2}}\left[\frac{\beta_1^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right)\right] + \frac{1}{3}\,\beta_2^{\frac{3}{2}}\left[\frac{\beta_2^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\right]}$$

$$= 2\lim_{h\to+\infty}\frac{\beta_0^{\frac{3}{2}}\left[\frac{\beta_0^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\right]}{\beta_2^{\frac{3}{2}}\left[\frac{\beta_2^{\frac{1}{2}}}{\eta} - \mathrm{sl}'\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\right]} = 2.$$

3. Conclusion

We have shown that the boundary value problem (2.2)–(2.3) for sufficiently large values of parameter h has exactly three nontrivial solutions. One of those solutions is symmetric with respect to $t = 0$ and two others are asymmetric as shown in Fig. 3. Both asymmetric solutions are the Nehari solutions.

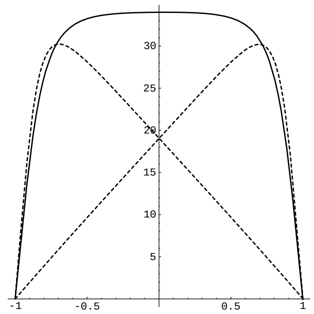

Figure 3. Three solutions of problem (2.2)–(2.3)

Computations show that the system (2.6) has exactly three solutions for $h > 1$. The respective three solutions of the boundary value problem looks like shown in Fig. 3. The value of the functional $H(x)$ for any of two asymmetric solutions is less than that of the symmetric solution.

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