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ON A FUČÍK TYPE SPECTRAL PROBLEM FOR THE SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH THE INTEGRAL BOUNDARY CONDITION

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Dedicated to Professor Jeff Webb on his jubilee

ABSTRACT. In this article we consider the equation $x'' = -\mu f(x^+) + \lambda g(x^-)$, where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, together with the boundary conditions $x(0) = 0$, $x(b) = \gamma \int_0^b x(s)\,ds$. We give description of a set of (μ, λ) such that the problem has a nontrivial solution.

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1. INTRODUCTION

In this paper we consider boundary value problem

$$x'' = -\mu f(x^+) + \lambda g(x^-), \tag{1}$$

$$x(0) = 0, \quad x(b) = \gamma \int_0^b x(s)\,ds, \tag{2}$$

where λ and μ are non-negative spectral parameters and $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. The functions $f, g \in C^1([0, +\infty) \to [0, +\infty))$ and $f(0) = g(0) = 0$.

Our research is motivated by:

- classical results on the Fučík spectrum where the boundary conditions are of the Dirichlet type ([7]);
- description of spectra for the problem (1), (2), where $\gamma = 0$ ([4])
- the results on spectra for the Fučík equation with the integral condition ([11], [12]).

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In what follows we give description of the spectrum of the problem (1), (2) in terms of time-map functions (the first zero functions) associated with f and g. We point out some interesting features of spectra and compare them with those for the Dirichlet type problems. The detailed analysis of the spectrum is made for the case of f and q being cubic functions. In this case description of spectra is made analytically and visualizations of spectra are obtained for selected values of a parameter γ.

2. REVIEW OF PREVIOUS RESULTS

2.1. **The classical Fučík spectrum.** The Fučík equation

$$x'' = -\mu x^+ + \lambda x^- \tag{3}$$

is a semi-linear equation with parameters. It does not satisfy the superposition principle (a sum of two solutions need not to be a solution), but it possesses the positive homogeneity property (if $x(t)$ is a solution of the equation then $kx(t)$ is a solution also provided that $k \geq 0$). A set of all (μ, λ) such that the problem (3),

$$x(0) = 0, \; x(b) = 0 \tag{4}$$

has a nontrivial solution is the Fučík spectrum. It is well known (see the left side of Fig. 1). It consists of the branches F_i^+ and F_i^-, where the integer i indicates how many zeros a respective solution $x(t)$ has in the interval $(0, 1)$ and the sign "+" refers to the initial value $x'(0) = 1$. Consequently the sign "−" means that a solution $x(t)$ satisfies the initial condition $x'(0) = -1$.

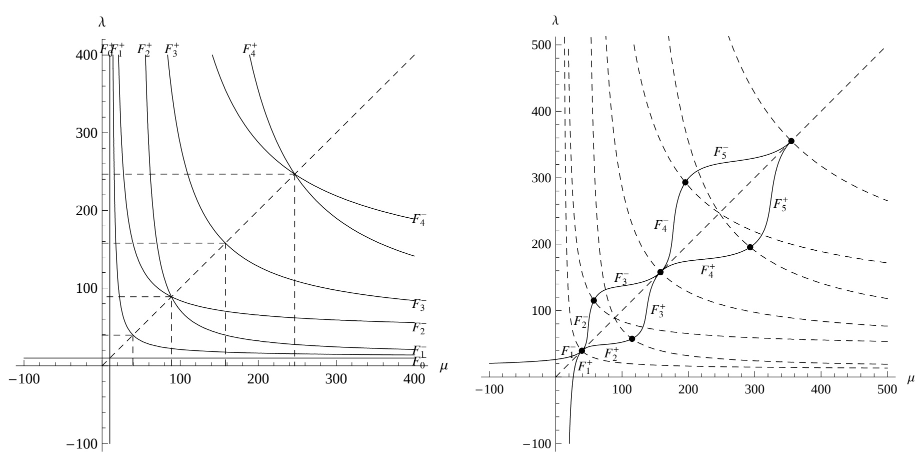

Figure 1. Spectra for the problems (3), (4) and (3), (5)

On the right side in Fig. 1 there is depicted the spectrum of the problem (3),

$$x(0) = 0, \quad \int_0^b x(s)\, ds = 0. \tag{5}$$

Both spectra differ significantly.

2.2. **The nonlinear Fučík problem.** The nonlinear (in the meaning that f and/or g may be nonlinear functions) problem (1), (4) was studied in [4], [5], [8] under the additional condition

$$|x'(0)| = \alpha. \tag{6}$$

Without the condition (6) the problem (1), (4) generally has continuous spectrum filling entire regions in the (μ, λ) plane. This condition was called *the normalization condition*.

The normalization condition is not needed in the problem (3), (4) since equation (3) possesses the positive homogeneity property and any positive multiple of a nontrivial solution is a solution also.

It was shown in [4], [5], [8] that properties of the spectrum for the problem (1), (4) depend entirely on the monotonicity properties of the functions $T_f(\mu, \alpha)$ and $T_g(\lambda, \alpha)$ (see the section below) considered as μ and λ functions respectively with α fixed.

Generally if $T_f(\mu, \cdot)$ and $T_g(\lambda, \cdot)$ are strictly monotonically decreasing then the spectra are similar (topologically equivalent) to the classical Fučík spectrum. If one or both of the above functions have local extrema then the spectrum may exhibit peculiar features: branches of the spectrum may contain multiple components, bounded or unbounded. In the latter case the asymptotics can be described in details.

2.3. **The integral conditions.** In the works [10], [11], [12] the Fučík equation was considered together with the integral conditions

$$x(a) = 0, \quad \int_a^b x(s)\, ds = 0 \tag{7}$$

or

$$x(a) = 0, \quad x(b) = \gamma \int_a^b x(s)\, ds. \tag{8}$$

The analytical and graphical descriptions of spectra were obtained for both boundary conditions.

The spectrum for the problem (1), (7) was obtained analytically and described graphically for the case $f = g = x^3$.

3. TECHNICAL BACKGROUND

In this section we introduce and describe functions and notions which are needed to investigate and formulate the results for the problem (1), (2).

First consider the Cauchy problem

$$x'' = -f(x), \quad x(0) = 0,\ x'(0) = \beta, \quad \beta > 0. \tag{9}$$

Denote a solution $x(t; \beta)$ and let $t_f(\beta)$ stand for the first zero of $x(t; \beta)$. The function $t_f(\beta)$ is called a time-map function ([9]). This function is defined and continuous if

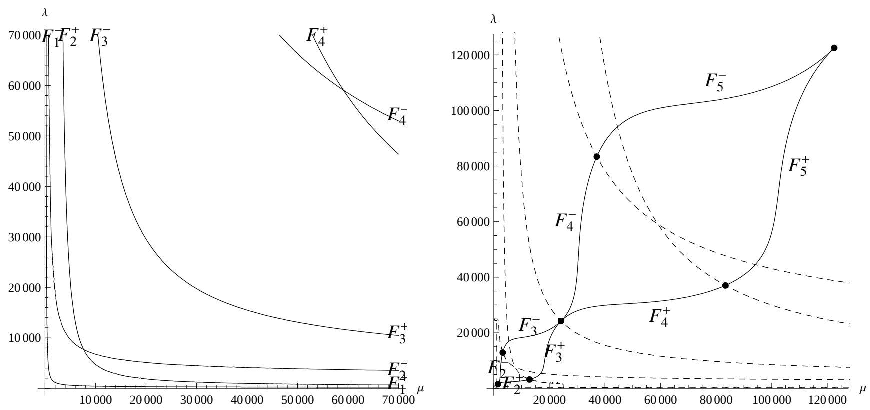

FIGURE 2. The spectra for the problems (16), (4) and (16), (17) for $|\alpha| = 1$ and $b = 1$.

$f(x)$ is a continuously differentiable function. If there is not the first zero for some β we set $t_f(\beta) = +\infty$.

Consider the problem

$$X'' = -\mu f(X), \quad X(0) = 0,\ X'(0) = \alpha, \quad \alpha > 0. \tag{10}$$

Proposition 3.1. *If $x(t;\beta)$ solves the problem* (9) *then $X(t;\mu,\alpha) = x(\sqrt{\mu}t; \frac{\alpha}{\sqrt{\mu}})$ solves the Cauchy problem* (10).

It can be verified directly that $x(\sqrt{\mu}t; \frac{\alpha}{\sqrt{\mu}})$ satisfies the equation and the initial conditions in (10).

Corollary 3.1. *If $t_f(\beta)$ is a time map for the problem* (9) *then*

$$T_f(\mu,\alpha) = \frac{1}{\sqrt{\mu}} t_f(\frac{\alpha}{\sqrt{\mu}}) \tag{11}$$

is a time map for the problem (10).

Proof. If $x(t;\beta)$ vanishes at $t_f(\beta)$ then $X(t;\mu,\alpha) = x(\sqrt{\mu}t; \frac{\alpha}{\sqrt{\mu}})$ vanishes at $\frac{1}{\sqrt{\mu}} t_f(\frac{\alpha}{\sqrt{\mu}})$.

Denote $s_f(t;\beta) = \int_0^t x(s;\beta)\,ds$ and $S_f(t;\mu,\alpha) = \int_0^t X(s;\mu,\alpha)\,ds$, where x and X are solutions of the Cauchy problems (9) and (10) respectively.

Corollary 3.2.

$$S_f(t;\mu,\alpha) = \frac{1}{\sqrt{\mu}} s_f(\sqrt{\mu}t; \frac{\alpha}{\sqrt{\mu}}). \tag{12}$$

The formula (12) can be verified using the rescaling argument.

Similar notation is introduced for a function g.

3.1. **Examples of functions t and s.** For equation $x'' + \omega^2 x = 0$ one has:

$$t(\alpha) = \frac{\pi}{\omega}, \quad s(t;\alpha) = \int_0^t \sin\omega s\, ds = \frac{\alpha}{\omega^2}[1 - \cos\omega t]. \tag{13}$$

The functions $T(\mu,\alpha)$ and $S(t;\mu,\alpha)$ for the problem

$$X'' + \mu\,\omega^2 X = 0, \quad X(0) = 0,\ X'(0) = \alpha > 0$$

are respectively $\dfrac{1}{\sqrt{\mu}}\dfrac{\pi}{\omega}$ and $\dfrac{\alpha}{\mu\,\omega^2}(1 - \cos\sqrt{\mu}\,\omega\, t)$.

For equation $x'' + 2x^3 = 0$ one has:

$$t(\alpha) = 2\frac{A}{\sqrt{\alpha}}, \quad s(t;\alpha) = \int_0^t \sqrt{\alpha}\ \mathrm{sl}\ \sqrt{\alpha} s\, ds = \frac{\pi}{4} - \arctan \mathrm{cl}\sqrt{\alpha} t, \tag{14}$$

where $A = \displaystyle\int_0^1 \frac{dt}{\sqrt{1 - t^4}}$. The computations yield

$$T(\mu,\alpha) = \mu^{-\frac{1}{4}}\alpha^{-\frac{1}{2}} \cdot 2A,$$
$$S(t;\mu,\alpha) = \frac{1}{\sqrt{\mu}}[\frac{\pi}{4} - \arctan \mathrm{cl}\,(\mu^{\frac{1}{4}}\alpha^{\frac{1}{2}} t)].$$

The functions $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$ are lemniscatic sine and cosine functions, the constant A for the lemniscatic sine is as the constant $\frac{\pi}{2}$ for the trigonometrical $\sin t$.

In the sequel A is the value of the above integral, $A \approx 1.311$.

More on lemniscatic functions can be learned from [13] as well as from modern sources [3]. It is important for calculations that lemniscatic functions can be expressed via the Jacobi elliptic functions.

4. DESCRIPTION OF BRANCHES OF THE SPECTRUM

Consider the problem (1), (2) together with the normalization condition (6). Suppose that also

$$\int_0^\infty f(x)dx = +\infty, \quad \int_0^\infty g(x)dx = +\infty. \tag{15}$$

These conditions ensure that solutions of the Cauchy problems (9) and the respective problem for a function g have zeros and therefore the functions t_f and t_g are finite. It is true also that solutions of these problems are symmetric in the intervals $(0, t_f(\beta))$ and $(0, t_g(\beta))$ with respect to the middle points of the intervals. Therefore solutions of the equation (1) which satisfy the initial conditions $x(0) = 0$, $x'(0) = \alpha$ satisfy also the condition $|x'(z)| = \alpha$ at any zero point.

In order to obtain the relations for finding points of the spectrum we consider solutions with different nodal structure.

4.1. **Solutions with different nodal structure.**

4.1.1. *Solutions without zeros.* Consider first solutions of the equation (1) which do not have zeros in the interval $(0, b)$. We distinguish between two cases, namely, $x'(0) = \alpha > 0$ or $x'(0) = -\alpha < 0$.

Define "positive" branch F_0^+ by the relation:

$$x(b;\alpha) = \gamma\, S_f(b;\mu,\alpha), \quad T_f(\mu,\alpha) \le b.$$

The "negative" branch F_0^- is defined by:

$$-y(b;\alpha) = -\gamma\, S_g(b;\lambda,\alpha), \quad T_g(\mu,\alpha) \le b,$$

where $y(t)$ is a solution of $y'' = -g(y),\ y(0) = 0,\ y'(0) = \alpha$. The above two relations provide sets of values of μ and λ which form the zero branches of the spectrum. This sets may be empty, of course, for instance, if $\gamma < 0$.

4.1.2. *Solutions with exactly one zero.* Let $\Theta := T_f(\mu,\alpha)$ and $\Phi := T_g(\lambda,\alpha)$.

The "positive" branch F_1^+ is defined by:

$$-y(b-\Theta;\alpha) = \gamma\,[S_f(\Theta;\mu,\alpha) - S_g(b-\Theta;\lambda,\alpha)],$$
$$\Theta < b, \quad \Theta + \Phi \ge b.$$

The "negative" branch is defined by:

$$x(b-\Phi;\alpha) = \gamma\,[-S_g(\Phi;\lambda,\alpha) + S_f(b-\Phi;\mu,\alpha)],$$
$$\Phi < b, \quad \Phi + \Theta \ge b.$$

Generally,

$$F_{2i-1}^+ : \frac{1}{\gamma}[-y(b-(i\Theta+(i-1)\Phi))] = iS_f(\Theta) - (i-1)S_g(\Phi) - S_g(b-(i\Theta+(i-1)\Phi)),$$
$$i\Theta + (i-1)\Phi < b, \quad i\Theta + i\Phi > b,$$
$$F_{2i-1}^- : \frac{1}{\gamma}[x(b-((i-1)\Theta+i\Phi))] = (i-1)S_f(\Theta) - iS_g(\Phi) + S_f(b-((i-1)\Theta+i\Phi)),$$
$$(i-1)\Theta + i\Phi < b, \quad i\Theta + i\Phi > b,$$
$$F_{2i}^+ : \frac{1}{\gamma}[x(b-i(\Theta+\Phi))] = iS_f(\Theta) - iS_g(\Phi) + S_f(b-i(\Theta+\Phi)),$$
$$i(\Theta+\Phi) < b, \quad i(\Theta+\Phi) + \Theta > b,$$
$$F_{2i}^- : \frac{1}{\gamma}[-y(b-i(\Theta+\Phi))] = iS_f(\Theta) - iS_g(\Phi) - S_g(b-i(\Theta+\Phi)),$$
$$i(\Theta+\Phi) < b, \quad i(\Theta+\Phi) + \Phi > b,$$
$$i = 1, 2, \dots$$

Theorem 4.1. *The spectrum for the problem* (1), (2) *consists of the branches given by the above relations, where* $\gamma \neq 0$.

If $\gamma = 0$ then the conditions (2) are of the Dirichlet type and the results in [4] cover this case.

5. CUBIC NONLINEARITIES

5.1. **Zero integral condition.** Consider the equation

$$-x'' = \mu(x^+)^3 - \lambda(x^-)^3, \quad \mu > 0, \quad \lambda > 0 \tag{16}$$

with the conditions

$$x(0) = 0, \quad \int_0^b x(s)ds = 0, \quad |x'(0)| = \alpha. \tag{17}$$

Theorem 5.1. *The Fučík spectrum for the problem* (16), (17) *consists of the branches given by*

$$F^+_{2i-1} = \Big\{(\mu,\lambda)\Big|\ \frac{i\pi}{2}\sqrt{\lambda} - \frac{(2i-1)\pi}{4}\sqrt{\mu} - \sqrt{\mu}\arctan\mathrm{cl}\,\big(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}}b + 2Ai-$$
$$-2Ai\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{\mu}}\big) = 0,\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}(i-1) < \frac{b}{2A},\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}i \geq \frac{b}{2A}\Big\},$$

$$F^+_{2i} = \Big\{(\mu,\lambda)\Big|\ \frac{(2i+1)\pi}{4}\sqrt{\lambda} - \frac{i\pi}{2}\sqrt{\mu} - \sqrt{\lambda}\arctan\mathrm{cl}\,\big(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}} + 2Ai-$$
$$-2Ai\sqrt{\alpha}\sqrt[4]{\frac{\mu}{\lambda}}\big) = 0,\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}i < \frac{b}{2A},\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}(i+1) \geq \frac{b}{2A}\Big\},$$

$$F^-_i \;\; = \Big\{(\mu,\lambda)\Big|(\lambda,\mu) \in F^+_i\Big\},$$

where $i = 1, 2, \ldots$.

Proof. The proof of this theorem is analogous to the proof of Theorem 5.2.

5.2. **Connections between the Dirichlet and the integral condition.** Consider the equation (16) with the conditions

$$x(0) = 0, \quad |x'(0)| = \alpha, \quad x(b) = \gamma\int_0^b x(s)ds, \quad \gamma \in \mathbb{R}. \tag{18}$$

The expressions of the branches of the spectrum for the problem (16), (18) are given in the next theorem.

Theorem 5.2. *The spectrum for the problem* (16), (18) *consists of the branches (if these branches exist for corresponding value of* γ*) given by*

$$F^+_0 \;\; = \Big\{(\mu,\lambda)\Big|\ \sqrt{\alpha}\sqrt[4]{\frac{2}{\mu}}\mathrm{sl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}b) = \gamma\sqrt{\frac{2}{\mu}}\big(\frac{\pi}{4} - \arctan\mathrm{cl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}b)\big),$$
$$0 < \mu \leq 2\big(\frac{2A}{b}\big)^4\Big\},$$

$$\begin{aligned}
F_{2i-1}^{+} &= \Big\{(\mu,\lambda)\Big|\ \sqrt{\mu}\gamma \arctan \mathrm{cl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}} - 2Ai\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{\mu}} + 2Ai) - \\
&-\frac{i\pi\sqrt{\lambda}}{2}\gamma + \frac{(2i-1)\pi\sqrt{\mu}}{4}\gamma + \sqrt[4]{\frac{\mu^2\lambda}{2}}\mathrm{sl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}} - 2Ai\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{\mu}} + \\
&+2Ai) = 0,\ \ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}(i-1) < \frac{b}{2A},\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}i \geq \frac{b}{2A}\Big\}, \\
F_{2i}^{+} &= \Big\{(\mu,\lambda)\Big|\ \sqrt{\lambda}\gamma \arctan \mathrm{cl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}} - 2Ai\sqrt{\alpha}\sqrt[4]{\frac{\mu}{\lambda}} + 2Ai) - \\
&-\frac{(2i+1)\pi\sqrt{\lambda}}{4}\gamma + \frac{i\pi\sqrt{\mu}}{2}\gamma + \sqrt[4]{\frac{\lambda^2\mu}{2}}\mathrm{sl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}} - 2Ai\sqrt{\alpha}\sqrt[4]{\frac{\mu}{\lambda}} + \\
&+2Ai) = 0,\ \ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}i < \frac{b}{2A},\ \sqrt[4]{\frac{2}{\mu}}i + \sqrt[4]{\frac{2}{\lambda}}(i+1) \geq \frac{b}{2A}\Big\}, \\
F_i^{-} &= \Big\{(\mu,\lambda)\Big|(\lambda,\mu) \in F_i^{+}\Big\},
\end{aligned}$$

where $i = 1, 2, \ldots$.

Proof. The idea of the proof of this theorem is the same as in the works [10] and [12].

Some comments follow.

First of all we obtain the expression for F_0^{+}. Let us suppose that the solution without zeroes in the interval $(0, b)$ exists and $x'(0) = \alpha > 0$. In this case the problem (16), (18) reduces to the eigenvalue problem

$$-x'' = \mu x^3, \quad x(0) = 0, \quad x(b) = \gamma \int_0^b x(s)ds. \tag{19}$$

A solution which satisfies the initial conditions $x(0) = 0$, $x'(0) = \alpha$ is

$$x(t) = \sqrt{\alpha}\sqrt[4]{\frac{2}{\mu}}\mathrm{sl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}t).$$

We use this expression to satisfy the integral boundary condition in (19).

It is known from the work [11] (and [1]) that

$$\int_0^t \mathrm{sl}\, s ds = \frac{\pi}{4} - \arctan \mathrm{cl}\, t.$$

In view of

$$x(b) = \sqrt{\alpha}\sqrt[4]{\frac{2}{\mu}}\mathrm{sl}\,(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}b)$$

and

$$\int_0^b x(s)ds = \sqrt{\frac{2}{\mu}}(\frac{\pi}{4} - \arctan \mathrm{cl}\,\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}b)$$

we obtain the expression for F_0^{+}.

The idea of the proof for other branches is similar. We consider the eigenvalue problems in the intervals between two consecutive zeroes of the solution and use the conditions of the solutions for these problems.

For example, we will prove this theorem for F_1^+.

Suppose that $(\mu, \lambda) \in F_1^+$ and let $x(t)$ be the corresponding nontrivial solution of the problem (16), (18). The solution has only one zero in $(0,1)$ and $x'(0) = \alpha > 0$. Let this zero be denoted by τ.

Consider a solution of the problem (16), (18) in the interval $(0, \tau)$. We obtain that the problem (16), (18) in this interval reduces to the eigenvalue problem $-x'' = \mu x^3$ with boundary conditions $x(0) = x(\tau) = 0$. Since $x(t) = \sqrt{\alpha}\sqrt[4]{\frac{2}{\mu}}\mathrm{sl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}t\right)$ in the interval $(0, \tau)$ and $x(\tau) = 0$ we obtain that $\tau = 2A\sqrt{\frac{1}{\alpha}}\sqrt[4]{\frac{2}{\mu}}$.

It follows that

$$\int_0^\tau \sqrt{\alpha}\sqrt[4]{\frac{2}{\mu}}\mathrm{sl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\mu}{2}}s\right)ds = \frac{\pi}{2}\sqrt{\frac{2}{\mu}}. \tag{20}$$

Now consider a solution of the problem (16), (18) in the interval (τ, b). We obtain the problem $-x'' = \lambda x^3$ with boundary condition $x(\tau) = 0$ in this interval. Since $x(t) = -\sqrt{\alpha}\sqrt[4]{\frac{2}{\lambda}}\mathrm{sl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}}(t - 2A\sqrt{\frac{1}{\alpha}}\sqrt[4]{\frac{2}{\mu}})\right)$ in the interval (τ, b) we obtain that

$$x(b) = -\sqrt{\alpha}\sqrt[4]{\frac{2}{\lambda}}\mathrm{sl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}}b - 2A\sqrt[4]{\frac{\lambda}{\mu}}\right) \tag{21}$$

and

$$\int_\tau^b (-\sqrt{\alpha})\sqrt[4]{\frac{2}{\lambda}}\mathrm{sl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}}t - 2A\sqrt[4]{\frac{\lambda}{\mu}}\right)ds =$$

$$= \sqrt{\frac{2}{\lambda}}\Big(\arctan\mathrm{cl}\left(\sqrt{\alpha}\sqrt[4]{\frac{\lambda}{2}}b - 2A\sqrt[4]{\frac{\lambda}{\mu}}\right) - \frac{\pi}{4}\Big). \tag{22}$$

Using (20), (21) and (22) in the condition $x(b) = \gamma\int_0^b x(s)ds$ we obtain the expression for F_1^+.

Considering the solution of the problem (16), (18) it is easy to prove that $0 < 2A\sqrt[4]{\frac{2}{\mu}} < b \le 2A\sqrt[4]{\frac{2}{\mu}} + 2A\sqrt[4]{\frac{2}{\lambda}}$.

The proof for other branches is analogous.

Several first branches of the spectrum to the problem (16), (18) for different γ values and $b = 1$, $|\alpha| = 1$ are depicted in Figure 3, the dashed curve is the spectrum for the problem with Dirichlet conditions, the positive and the negative branches of the spectrum for the problem (16), (18) are indicated.

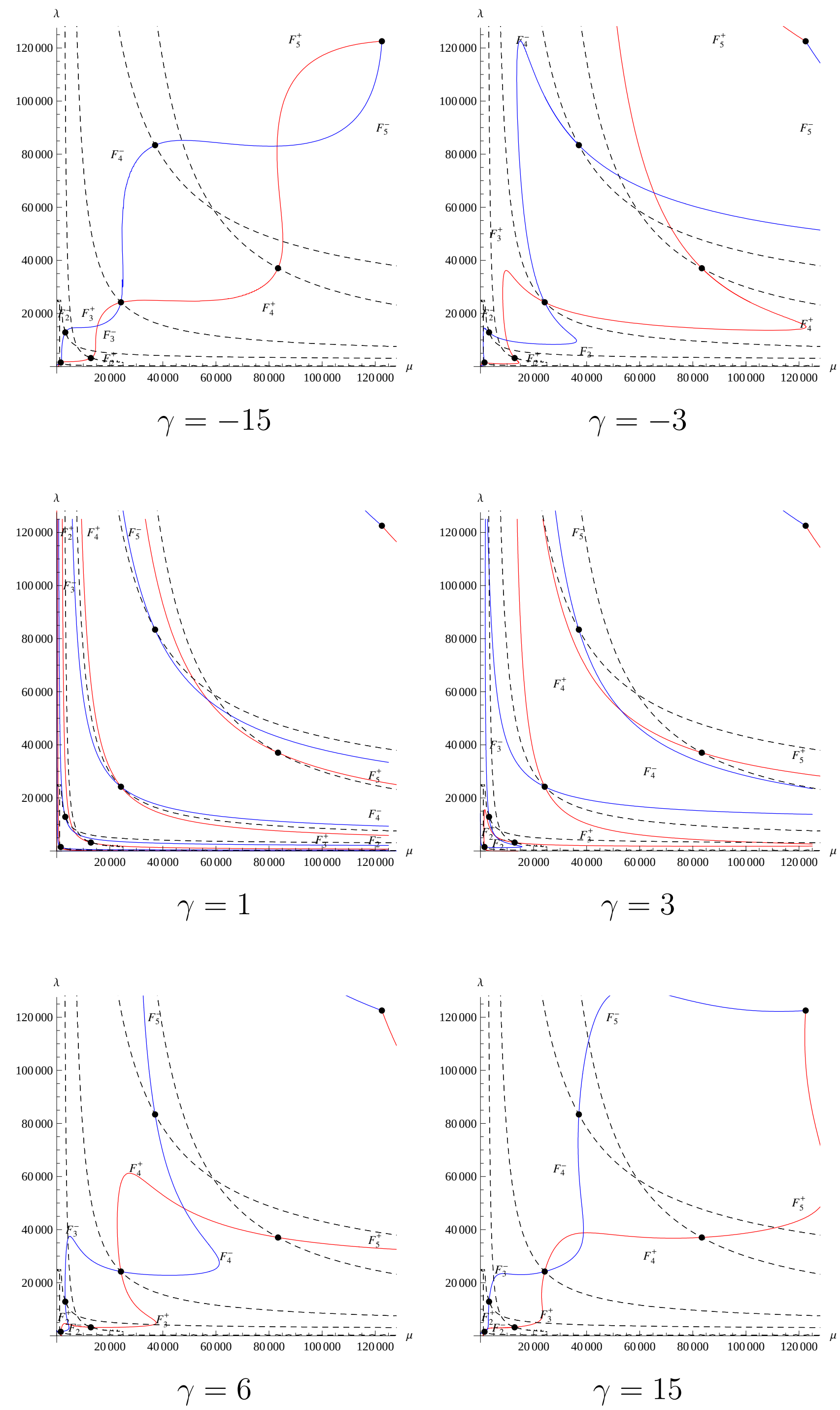

FIGURE 3. The spectrum of the problem (16), (18) for different values of γ, $b = 1$, $|\alpha| = 1$.

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