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Research Article

Unexpected Solutions of the Nehari Problem

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The Nehari characteristic numbers $\lambda_n(a,b)$ are the minimal values of an integral functional associated with a boundary value problem (BVP) for nonlinear ordinary differential equation. In case of multiple solutions of the BVP, the problem of identifying of minimizers arises. It was observed earlier that for nonoscillatory (positive) solutions of BVP those with asymmetric shape can provide the minimal value to a functional. At the same time, an even solution with regular shape is not a minimizer. We show by constructing the example that the same phenomenon can be observed in the Nehari problem for the fifth characteristic number $\lambda_n(a,b)$ which is associated with oscillatory solutions of BVP (namely, with those having exactly four zeros in (a,b)).

1. Introduction

The variational theory of eigenvalues in Sturm-Liouville problems for linear ordinary differential equations provides variational interpretation of eigenvalues which emerge as minima of some quadratic functionals being considered with certain restrictions [1].

As to nonlinear boundary value problems for ordinary differential equations, the Nehari theory of characteristic values provides some analogue of the linear theory. The Nehari theory deals in particular with superlinear differential equations of the form

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0. \tag{1}$$

The Nehari numbers $\lambda_n(a,b)$, by definition, are minimal values of the functional

$$H(x) = \int_a^b \left[x'^2(t) - (1+\varepsilon)^{-1} q(t)\, x^{2+2\varepsilon}(t)\right] dt \tag{2}$$

over the set Γ_n of all functions $x(t)$, which are (1) continuous and piecewise continuously differentiable in $[a,b]$; (2) there exist numbers a_ν such that $a = a_1 < \cdots < a_{n-1} = b$ and $x(a_\nu) = 0$ in any a_ν; (3) in any $[a_{\nu-1}, a_\nu]$, $x(t) \not\equiv 0$ and

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\, dt = \int_{a_{\nu-1}}^{a_\nu} q(t)\, x^2 |x|^{2\varepsilon} dt. \tag{3}$$

It was proved in [2] (see also [3]) that minimizers in the above variational problem are C^2-solutions of the boundary value problem

$$\begin{gathered} x'' = -q(t)|x|^{2\varepsilon}x, \qquad x(a) = x(b) = 0, \\ x(t) \text{ has exactly } n-1 \text{ zeros in } (a,b). \end{gathered} \tag{4}$$

Putting (3) into (2) one gets

$$\begin{aligned} \lambda_n(a,b) &= \min_{x\in\Gamma_n} H(x) \\ &= \frac{\varepsilon}{1+\varepsilon}\int_a^b q(t)\, x^{2+2\varepsilon}(t)\, dt \\ &= \frac{\varepsilon}{1+\varepsilon}\int_a^b x'^2(t)\, dt, \end{aligned} \tag{5}$$

where $x(t)$ is an appropriate solution of the BVP (4). The BVP (4) may have multiple solutions but not all of them are minimizers.

It appears that in order to detect $\lambda_n(a,b)$ it is sufficient to consider solutions of the boundary value problem (4).

Z. Nehari posed the question is it true that there is only one minimizer associated with $\lambda_n(a,b)$. It was shown implicitly in [4] that there may be multiple minimizers associated

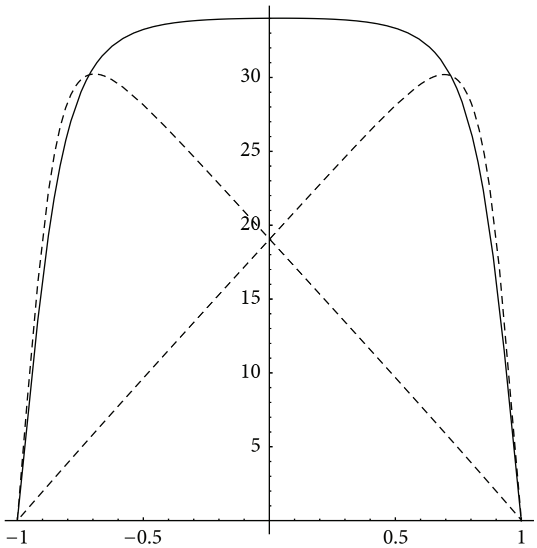

FIGURE 1: Three solutions of the BVP from [5].

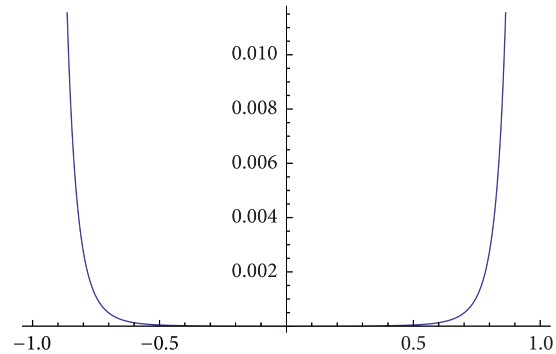

FIGURE 2: The function $q(t)$ in (9) for $h = 10$ and $\eta = 11$.

with the first characteristic number $\lambda_1(a, b)$. These minimizers are positive solutions of the problem (4) ($n = 1$).

Later in the work by the authors [5] the example was constructed showing three solutions of the BVP (4). They are depicted in Figure 1.

Two of these solutions are asymmetric and one is an even function. Surprisingly, two asymmetric solutions are the minimizers.

The same phenomenon was observed later by Kajikiya [6] who studied "the Emden-Fowler equation whose coefficient is even in the interval $(-1, 1)$ under the Dirichlet boundary condition." It was proved that if the density of the coefficient function is thin in the interior of $(-1, 1)$ and thick on the boundary, then a least energy solution is not even. Therefore, the equation has at least three positive solutions: the first one is even, the second one is a non-even least energy solution $u(t)$, and the third one is the reflection $u(-t)$. Similar phenomena were discussed in [7, 8].

In this note, we confirm this phenomenon for the characteristic value λ_5. Solutions for this characteristic value have exactly four zeros in an open interval (a, b). We construct the example and provide all the calculations and visualizations.

2. Preliminaries

2.1. Conventions and Definitions. The problem of finding characteristic values $\lambda_n(a, b)$ is called the Nehari problem. A function $x(t)$ that supplies minimal value in the Nehari problem will be called the Nehari solution. Nehari solutions associated with the equation, the interval (a, b), and the number n, all are solutions of the respective boundary value problem (4).

2.2. Auxiliary Functions. In our constructions below we use the auxiliary functions called the lemniscatic sine $\mathrm{sl}\, t$ and cosine $\mathrm{cl}\, t$. These functions can be introduced as solutions of the Cauchy problems, respectively, as follows:

$$x(0) = 0, \quad x'(0) = 1, \quad x(0) = 1, \quad x'(0) = 0 \tag{6}$$

for the equation $x'' = -2x^3$. These functions are much like the usual sine and cosine functions, but they are not the derivatives of each other. Instead the following holds:

$$\mathrm{sl}'t = \mathrm{cl}\, t\left(1 + \mathrm{sl}^2 t\right), \qquad \mathrm{cl}'t = -\mathrm{sl}\, t\left(1 + \mathrm{cl}^2 t\right). \tag{7}$$

A complete list of formulas for the lemniscatic functions can be found in [9]. The lemniscatic functions can be handled symbolically by the Wolfram Mathematica program using the representation in terms of the built-in Jacobi functions.

3. Construction of the Equation

Consider the interval $[-1, 1]$. Define the piecewise linear function

$$\xi(t) = \begin{cases} ht + \eta, & t \in [-1, 0], \\ -ht + \eta, & t \in [0, 1], \end{cases} \tag{8}$$

where $\eta = h + 1$ and $h > 0$ is a selected number. Define $q(t) = 2/\xi(t)^6$. The function $q(t)$ (depicted in Figure 2) is U-shaped function "thin" in the middle of the interval $[-1, 1]$.

Consider equation

$$x'' = -q(t)\, x^3, \quad t \in (-1, 1), \tag{9}$$

together with the boundary conditions

$$x(-1) = 0, \quad x(1) = 0. \tag{10}$$

Consider the Cauchy problems

$$\begin{gathered} x_1'' = -\frac{k}{(ht + \eta)^6}\, x_1^3, \\ x_1(-1) = 0, \quad x_1'(-1) = \beta, \quad t \in (-1, 0); \end{gathered} \tag{11}$$

$$\begin{gathered} x_2'' = -\frac{k}{(-ht + \eta)^6}\, x_2^3, \\ x_2(1) = 0, \quad x_2'(1) = -\gamma, \quad t \in (0, 1). \end{gathered} \tag{12}$$

Let

$x_1(t, \beta)$ be a solution of (11) in $[-1, 0]$,
$x_2(t, \gamma)$ a solution of (12) in $[0, 1]$.

Then, the function

$$x(t) = \begin{cases} x_1(t, \beta), & \text{if } -1 \le t \le 0, \\ x_2(t, \gamma), & \text{if } 0 \le t \le 1 \end{cases} \tag{13}$$

is a C^2-solution of the problem (9) and the problem (10) if the continuity and smoothness conditions

$$x_1(0, \beta) = x_2(0, \gamma), \qquad x_1'(0, \beta) = x_2'(0, \gamma) \tag{14}$$

are satisfied. The problems (11) and (12) can be solved explicitly as

$$x_1(t, \beta) = \beta^{1/2}(ht + \eta) \cdot \mathrm{sl}\left(\beta^{1/2}\frac{t+1}{ht+\eta}\right), \quad t \in [-1, 0], \tag{15}$$

$$x_2(t, \gamma) = -\gamma^{1/2}(-ht + \eta) \cdot \mathrm{sl}\left(\gamma^{1/2}\frac{t-1}{-ht+\eta}\right), \quad t \in [0, 1], \tag{16}$$

where

$$\begin{gathered} \beta = x_1'(-1) > 0, \\ -\gamma = x_2'(1). \end{gathered} \tag{17}$$

The derivatives can be computed as

$$\begin{aligned} x_1'(t; \beta) = {} & \beta^{1/2} h \cdot \mathrm{sl}\left(\beta^{1/2}\frac{t+1}{ht+\eta}\right) \\ & + \beta\frac{-h+\eta}{ht+\eta} \cdot \mathrm{sl}'\left(\beta^{1/2}\frac{t+1}{ht+\eta}\right). \end{aligned} \tag{18}$$

In order to get an explicit formula for a solution of the BVP (9) and (10), one has to solve a system of two equations with respect to (β, γ):

$$x_1(0; \beta) = x_2(0; \gamma), \qquad x_1'(0; \beta) = x_2'(0; \gamma). \tag{19}$$

This system after replacements and simplifications is

$$\begin{gathered} \beta^{1/2} \cdot \mathrm{sl}\left(\frac{\beta^{1/2}}{\eta}\right) = \gamma^{1/2} \cdot \mathrm{sl}\left(\frac{\gamma^{1/2}}{\eta}\right), \\ \beta^{1/2} h \cdot \mathrm{sl}\left(\frac{\beta^{1/2}}{\eta}\right) + \frac{\beta}{\eta} \cdot \mathrm{sl}'\left(\frac{\beta^{1/2}}{\eta}\right) \\ = -\gamma^{1/2} h \cdot \mathrm{sl}\left(\frac{\gamma^{1/2}}{\eta}\right) - \frac{\gamma}{\eta} \cdot \mathrm{sl}'\left(\frac{\gamma^{1/2}}{\eta}\right). \end{gathered} \tag{20}$$

In new variables $u := \beta^{1/2}/\eta$ and $v := \gamma^{1/2}/\eta$, the system takes the form

$$\begin{gathered} u \,\mathrm{sl}\, u = v \,\mathrm{sl}\, v, \\ hu \,\mathrm{sl}\, u + u^2 \mathrm{sl}' u = -hv \,\mathrm{sl}\, v - v^2 \mathrm{sl}' v, \quad h > 0. \end{gathered} \tag{21}$$

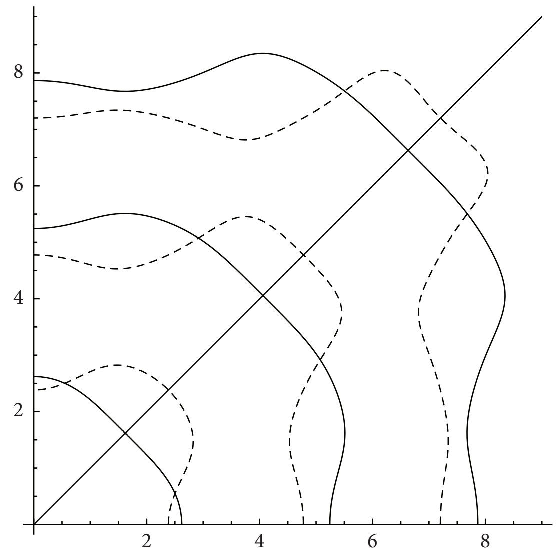

FIGURE 3: Zeros of $\Phi(u, v)$ (solid line) and $\Psi(u, v)$ (dashed line), $h = 10$.

Notice that if a solution (u_0, v_0) of the system (21) is known, then a solution $x(t)$ of the BVP (9) and (10) can be constructed such that

$$x'(-1) = \beta = u_0^2\eta^2, \qquad x'(1) = -\gamma = -v_0^2\eta^2. \tag{22}$$

If we introduce the functions

$$\begin{gathered} \Phi(u, v) = u \,\mathrm{sl}\, u - v \,\mathrm{sl}\, v, \\ \Psi(u, v) = hu \,\mathrm{sl}\, u + u^2 \mathrm{sl}' u + hv \,\mathrm{sl}\, v + v^2 \mathrm{sl}' v, \end{gathered} \tag{23}$$

the system (21) can be rewritten as

$$\begin{gathered} \Phi(u, v) = 0, \\ \Psi(u, v) = 0. \end{gathered} \tag{24}$$

Zeros of the functions $\Phi(u, v)$ and $\Psi(u, v)$ in the square $Q = \{(u, v) : 0 \le u, v \le 9\}$ are depicted in Figure 3. Notice that a set of zeros of Φ consists of the diagonal $u = v$ and the "wings."

The intersection points of smaller "hat" with the zero set of $\Phi(u, v)$ reflect three solutions of the problem (9) and (10) depicted in Figure 1. The cross-point on the bisectrix relates to the even solution, and two symmetric cross-points on the "wings" relate to the remaining two solutions of asymmetric shape. The latter two solutions are "unexpected" minimizers (or, as in [6], "non-even least energy solutions").

It was proved in [5, Proposition 1] that for h sufficiently large there are exactly three cross-points on a smaller "hat" (probably, for any $h > 1$). The similar proof can be conducted for the middle "hat" in Figure 3. There are exactly three cross-points (and exactly three solutions of the system (24)) that give rise to solutions of the problem (9) and the problem (10) that have exactly two zeros in the interval $(-1, 1)$. The respective values of the Nehari functional $H(x)$ (5) were calculated and the result is the same: the even solution

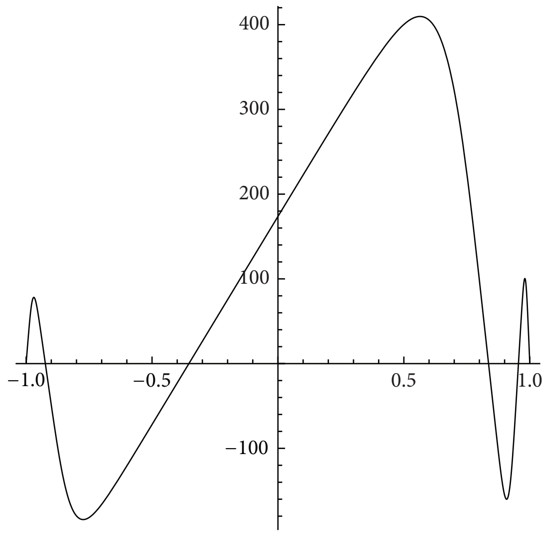

Figure 4: $x'(-1) = 3666.80$.

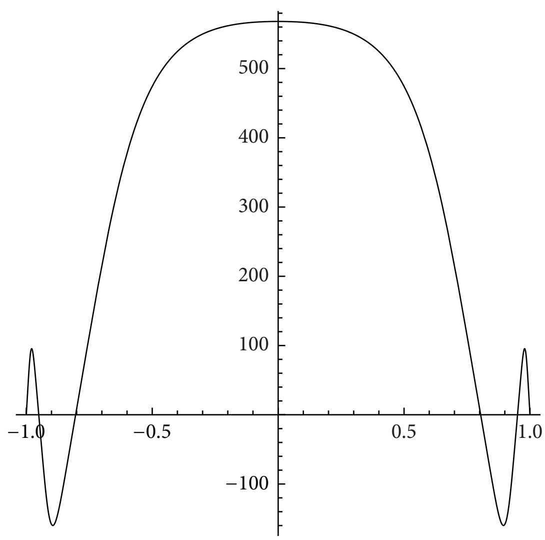

Figure 5: $x'(-1) = 6275.29$.

supplies $H(x_{\text{even}}) = 505549$, the two solutions of asymmetric shape supply the value $H(x_{\text{asym}}) = 332861$.

Therefore, we confirm the phenomenon observed in [5, 6] also for oscillatory (with exactly two zeros in $(-1, 1)$) solutions.

3.1. Nehari Solutions with Four Internal Zeros. We study in details the case of the Nehari characteristic number $\lambda_5(-1, 1)$. Related solutions of the boundary value problem have exactly four zeros in the interval $(-1, 1)$. Solving the system (24) on the third (counting from the origin) "hat" provides us with values

$$u_1 = 3666.804, \qquad v_1 = 7135.523; \qquad u_2 = v_2 = 6275.287;$$
$$u_3 = v_1, \qquad v_3 = u_1. \tag{25}$$

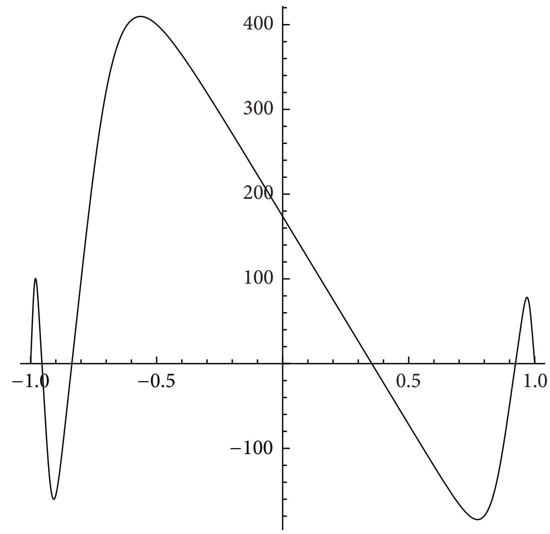

Figure 6: $x'(-1) = 7135.52$.

The respective solutions are known analytically through (15) and (16) and can be computed numerically. The second way yields the three graphs depicted in Figures 4, 5 and 6.

In order to detect the Nehari solutions, we compute the expression

$$\frac{\varepsilon}{1+\varepsilon} \int_a^b q(t)\, x^{2+2\varepsilon}(t)\, dt \tag{26}$$

in (5) for the three solutions depicted in Figure 4 to Figure 6. Recall that $\varepsilon = 1$, $a = -1$, and $b = 1$.

Let I_i be the numerical value of the above expression for solutions defined by the initial data $x(-1) = 0$, $x'(-1) = (u_i\eta)^2$, $i = 1, 2, 3$.

One gets that $I_1 = I_3 = 1968611.835$. The integral over the even solution is $I_2 = 2582181.527$. The Nehari solutions are those of asymmetric shape.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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