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Remarks on lemniscatic functions

Armands Gritsans[1], **Felix Sadyrbaev**[2]

[1] Department of Natural Sciences and Mathematics, Daugavpils University,
email: `arminge@dau.lv`

[2] Institute of Mathematics and Computer Science, University of Latvia,
email: `felix@cclu.lv`

A set of formulae is provided for the functions $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$ (known as the lemniscatic functions), which solve the differential system $\frac{x'}{1+x^2} = y$, $\frac{y'}{1+y^2} = -x$, as well as the Emden - Fowler equation $x'' = -2x^3$. We discuss similarity of the theory of the lemniscatic functions and that for elementary trigonometric functions and produce a set of formulae which are similar to those for $\sin t$ and $\cos t$. The addition theorem for $\mathrm{sl}\, t$ is given in various forms, some of them seem to be new. The theory of the Jacobian elliptic functions is used.

Key words: Lemniscatic functions, Jacobian elliptic functions, addition formulae

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Let us recall that the usual trigonometric functions can be introduced by considering the differential system

$$\begin{cases} x' = y, \\ y' = -x, \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{1}$$

Multiply the first equation by $2x$, the second one by $2y$ and sum up the both equations. One gets

$$d(x^2 + y^2) = 0$$

or

$$x^2(t) + y^2(t) = 1, \tag{2}$$

if taking into account the initial conditions in (1).

The relation (2) shows that the functions x and y define a unit circle.

It follows from (1) that

$$x'' = -x, \tag{3}$$

$$x(0) = 0, \quad x'(0) = 1.$$

Since the equation (3) is autonomous, any function $x(\alpha + t)$, where α is a constant, is also a solution of (3). The functions $x(t)$ and $y(t)$ are linearly independent solutions of (3) because the Wronskian

$$\det\begin{pmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{pmatrix} = \det\begin{pmatrix} x(t) & y(t) \\ y(t) & -x(t) \end{pmatrix} = -x^2(t) - y^2(t) = -1 \neq 0.$$

Then by properties of linear second order differential equations

$$x(\alpha + t) = C_1 x(t) + C_2 y(t), \tag{4}$$

where C_1 and C_2 are some constants to be found. Set $t = 0$. Then $x(\alpha) = C_1 x(0) + C_2 y(0) = C_2$. Since

$$x'(\alpha + t) = y(\alpha + t) = C_1 x'(t) + C_2 y'(t) = C_1 y(t) - C_2 x(t),$$

one obtains that

$$y(\alpha) = C_1 y(0) - C_2 x(0) = C_1.$$

Thus

$$x(\alpha + t) = y(\alpha) x(t) + x(\alpha) y(t). \tag{5}$$

The relation (5) is the so called *addition theorem* for the function $x(t)$ or simply the usual formula for a sine of two arguments $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

Any other important property of $\sin t$ and $\cos t$ can be derived from the differential system (1) (see [4], for example).

1. Nonlinear sine-like functions

We wish to use now the scheme of the previous section in order to treat the nonlinear differential system

$$\begin{cases} \dfrac{x'}{1+x^2} = y, \\ \dfrac{y'}{1+y^2} = -x, \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{6}$$

Multiply the first equation by $2x$, the second one by $2y$ and sum up the both equations. One gets then

$$d\Big(\ln\big[(1+x^2)(1+y^2)\big]\Big) = 0$$

or

$$\ln\big[(1+x^2)(1+y^2)\big] = const,$$

which, in its turn, gives

$$(1+x^2)(1+y^2) = 2,$$

taking into account the initial conditions in (6). The latter expression may be rewritten as

$$x^2(t) + x^2(t)y^2(t) + y^2(t) = 1. \tag{7}$$

It follows from (7) that

$$x^2(t) = \frac{1 - y^2(t)}{1 + y^2(t)} \tag{8}$$

and

$$y^2(t) = \frac{1 - x^2(t)}{1 + x^2(t)}. \tag{9}$$

The relation (7) defines a closed planar curve and provides an analogue of the unit circle (2).

We cannot use arguments of the previous section to deduce an addition theorem for the functions $x(t)$ and $y(t)$, defined by (6), because the differential system of (6) is nonlinear and does not allow the representation (4).

2. Lemniscatic functions

Let us rewrite the differential equations in (6) in the form

$$\begin{cases} x' = y\,(1 + x^2), \\ y' = -x\,(1 + y^2) \end{cases} \tag{10}$$

and differentiate the system (10). One obtains by using the relations (10) and (9) that

$$\begin{aligned} x'' &= y'(1 + x^2) + y \cdot 2xx' = -x(1 + x^2)(1 + y^2) + 2xy \cdot x' \\ &= -2x\,[1 - y\,x'] = -2x\,[1 - y^2(1 + x^2)] = -2x\left[1 - \tfrac{1-x^2}{1+x^2}(1 + x^2)\right] \\ &= -2\,x^3. \end{aligned}$$

It follows similarly, by virtue of (10) and (8), that

$$\begin{aligned} y'' &= -x'(1 + y^2) - x \cdot 2yy' = -y(1 + x^2)(1 + y^2) - 2xy \cdot y' \\ &= -2y\,[1 + x\,y'] = -2y\,[1 - x^2(1 + y^2)] = -2y\left[1 - \tfrac{1-y^2}{1+y^2}(1 + y^2)\right] \\ &= -2\,y^3. \end{aligned}$$

So it turns out that $x(t)$ and $y(t)$ are solutions of the same nonlinear second order differential equation

$$u'' = -2\,u^3, \tag{11}$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$ respectively.

Solutions of (11) satisfy the relations

$$u'^2 + u^4 = const.$$

Taking into account the initial conditions one gets that $x(t)$ and $y(t)$ satisfy the equality

$$u'^2 + u^4 = 1.$$

Then
$$\frac{du}{dt} = \pm\sqrt{1-u^4}$$
and the functions $x(t)$ and $y(t)$ can be expressed in the form
$$t = \int_0^{x(t)} \frac{ds}{\sqrt{1-s^4}} \tag{12}$$
and
$$t = \int_{y(t)}^{1} \frac{ds}{\sqrt{1-s^4}} \tag{13}$$
for $t \in [0, A]$, where $A := \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. The functions defined by the integral relations (12) and (13) are known as *the lemniscatic functions* [5, § 22.8]. So $x(t)$ and $y(t)$ can be identified with $\operatorname{sl} t$ and $\operatorname{cl} t$ respectively (the notation $\operatorname{sl} t$ and $\operatorname{cl} t$ for the lemniscatic functions was introduced by C.F. Gauss).

REMARK 1. *The usual* $\sin t$ *and* $\cos t$ *functions can be introduced in the same manner, namely, as* $t = \int_0^{\sin t} \frac{ds}{\sqrt{1-s^2}}$ *and* $t = \int_{\cos t}^{1} \frac{ds}{\sqrt{1-s^2}}$.

3. Jacobian elliptic functions

Let us remind basic properties of the Jacobian elliptic functions. The main three of them are $\operatorname{sn}(t;k)$, $\operatorname{cn}(t;k)$ and $\operatorname{dn}(t;k)$. They can be introduced as respective solutions of the (nonlinear) differential system
$$\begin{cases} x_1' = x_2 x_3, \\ x_2' = -x_1 x_3, \\ x_3' = -k^2 x_1 x_2, \quad 0 < k^2 < 1, \end{cases} \tag{14}$$
subject to the initial conditions
$$x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = 1.$$

The functions $\operatorname{sn}(t;k)$ and $\operatorname{cn}(t;k)$ are periodic $4K$-periodic and $\operatorname{dn}(t;k)$ is $2K$-periodic, where
$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}}.$$

The Jacobian elliptic functions satisfy the following basic relations [1, Ch. VII, § 1]:
$$\operatorname{sn}^2 t + \operatorname{cn}^2 t = 1, \quad k^2 \operatorname{sn}^2 t + \operatorname{dn}^2 t = 1, \tag{15}$$

which, in turn, imply

$$\operatorname{dn}^2 t - k^2 \operatorname{cn}^2 t = k_1^2,$$

where

$$k_1 = \sqrt{1-k^2}.$$

The functions $\operatorname{cn}(t;k)$ and $\operatorname{dn}(t;k)$ are even and $\operatorname{sn}(t;k)$ is odd.

The addition theorems for the Jacobian elliptic functions are known, namely [3, P. 753-765]:

$$\begin{aligned}
\operatorname{sn}(u+v) &= (\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1},\\
\operatorname{cn}(u+v) &= (cnu \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1},\\
\operatorname{dn}(u+v) &= (\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1}.
\end{aligned}$$

Other useful relations involving the Jacobian elliptic functions are:

$$\operatorname{sn}(t+K) = \frac{\operatorname{cn} t}{\operatorname{dn} t}, \quad \operatorname{cn}(t+K) = -k_1 \frac{\operatorname{sn} t}{\operatorname{dn} t}, \quad \operatorname{dn}(t+K) = k_1 \frac{1}{\operatorname{dn} t},$$

$$\operatorname{sn}(t+2K) = -\operatorname{sn} t, \quad \operatorname{cn}(t+2K) = -\operatorname{cn} t,$$

$$(\operatorname{sn} t)' = \operatorname{cn} t \operatorname{dn} t, \quad (\operatorname{cn} t)' = -\operatorname{sn} t \operatorname{dn} t, \quad (\operatorname{dn} t)' = -k^2 \operatorname{sn} t \operatorname{cn} t.$$

4. Relations between the Jacobian elliptic functions and the lemniscatic ones

Other nine Jacobian elliptic functions are introduced as some ratios involving the basic functions sn, cn and dn . functions above. In what follows we use also the function $\operatorname{sd}(t;k) = \frac{\operatorname{sn}(t;k)}{\operatorname{dn}(t;k)}$. It is known ([5, § 22.8]) that the lemniscatic functions can be expressed (at least in some neighborhood of $t = 0$) as [5, § 22.8]

$$\operatorname{sl} t = k \frac{\operatorname{sn} \frac{t}{k}}{\operatorname{dn} \frac{t}{k}}, \quad \operatorname{cl} t = \operatorname{cn} \frac{t}{k}, \quad k = \frac{1}{\sqrt{2}}. \tag{16}$$

In the sequel we derive the relations (16) on the whole real line $\mathbb{R}$ using only the definitions (11), (14) of the lemniscatic functions and properties of the Jacobian elliptic functions.

PROPOSITION 1. $\operatorname{sl} t = k \operatorname{sd} \frac{t}{k}$ *and* $\operatorname{cl} t = \operatorname{cn} \frac{t}{k}$ *for* $k = \frac{1}{\sqrt{2}}$.

Proof. Notice that $k = k_1 = \frac{1}{\sqrt{2}}$. Consider the functions $h(t) := k \operatorname{sd} t = k\frac{x_1(t)}{x_3(t)}$ and $g(t) := \operatorname{cn} t = x_2(t)$. It follows from (14) and (15) that

$$\begin{aligned}
h' &= \left(k\frac{x_1}{x_3}\right)' = k\frac{x_1' x_3 - x_1 x_3'}{x_3^2} = k\frac{(x_2x_3)x_3 - x_1(-k^2 x_1 x_2)}{x_3^2}\\
&= k\frac{x_2(x_3^2 + k^2 x_1^2)}{x_3^2} = k x_2\left(1 + \frac{k^2 x_1^2}{x_3^2}\right) = k\,g(1+h^2),
\end{aligned}$$

$$g' = -x_1x_3 = -\frac{x_1}{x_3}x_3^2 = -\frac{x_1}{x_3}(k^2x_2^2 + k^2) = -k^2\frac{x_1}{x_3}(1 + x_2^2) = -k\,h(1 + g^2).$$

The functions h and g satisfy also

$$h(0) = k\frac{x_1(0)}{x_3(0)} = 0, \quad g(0) = x_2(0) = 1.$$

Then the functions $h\left(\frac{t}{k}\right) = k\,\mathrm{sd}\,\frac{t}{k}$ and $g\left(\frac{t}{k}\right) = \mathrm{cn}\,\frac{t}{k}$ are solutions of the Cauchy problem (6). Solutions of the initial value problem (6) are unique since the right sides of the differential equations in (6) are polynomials and satisfy the Lipschitz condition in any bounded domain containing $\{(x,y) : |x| \leq 1,\ |y| \leq 1\}$. Hence the proof. □

The well known properties of $\mathrm{sl}\,t$ and $\mathrm{cl}\,t$ follow from the basic relations (16).

COROLLARY 1. *The function* $\mathrm{sl}\,t$ *is odd and the function* $\mathrm{cl}\,t$ *is even.*

COROLLARY 2. *The functions* $\mathrm{sl}\,t$ *and* $\mathrm{cl}\,t$ *are periodic with the minimal period of* $4A$, *where*

$$A = \int\limits_0^1 \frac{ds}{\sqrt{1-s^4}}\,. \tag{17}$$

COROLLARY 3. *The reduction formulae*

$$\mathrm{sl}(t + A) = \mathrm{cl}(t) \quad \textit{and} \quad \mathrm{cl}(t + A) = -\,\mathrm{sl}(t) \tag{18}$$

are valid.

REMARK 2. *Various reduction formulae can be derived for the functions* $\mathrm{sl}\,t$ *and* $\mathrm{cl}\,t$ *likely as in the case of the elementary functions* $\sin t$ *and* $\cos t$. *A constant* A *serves as the substitution for* $\pi/2$.

PROPOSITION 2. *The following relations are valid for any* $t \in \mathbb{R}$:

$$\mathrm{sl}'(t) = \mathrm{cl}(t)(1 + \mathrm{sl}^2(t)), \quad \mathrm{cl}'(t) = -\,\mathrm{sl}(t)(1 + \mathrm{cl}^2(t)),$$

$$\mathrm{sl}'^2(t) + \mathrm{sl}^4(t) = 1, \quad \mathrm{cl}'^2(t) + \mathrm{cl}^4(t) = 1, \quad \mathrm{sl}^2(t) + \mathrm{sl}^2(t)\,\mathrm{cl}^2(t) + \mathrm{cl}^2(t) = 1,$$

$$\mathrm{sl}(t + A) = \mathrm{cl}(t), \quad \mathrm{cl}(t + A) = -\,\mathrm{sl}(t), \ \textit{where}\ A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}\,.$$

Proof. Proofs can be found in [6, Propositions 7.4, 7.5 and 7.6, Corollary 7.3]. □

5. Summation results

The addition theorem for the lemniscatic functions was obtained by L. Euler in the integral form (for historical remarks one may consult [2, Sec. 2.3]). Let us mention that various forms of the sum formulae can be obtained directly from those for the Jacobian elliptic functions.

PROPOSITION 3.

$$\mathrm{sl}\,(\alpha+\beta)=\frac{\mathrm{sl}(\alpha)\,\mathrm{cl}(\beta)+\mathrm{cl}(\alpha)\,\mathrm{sl}(\beta)}{1-\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}\,. \tag{19}$$

PROPOSITION 4.

$$\mathrm{cl}\,(\alpha+\beta)=\frac{\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)-\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)}{1+\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}\,. \tag{20}$$

For the proofs one may consult [6].

The alternative forms of addition theorems are given below. Investigations of the sum formula for $\mathrm{sl}\,t$ go back to Fagnano and L. Euler [2, §§ 2.1, 2.2, 2.3]. The sum formula was obtained rather in the form

$$\mathrm{sl}(\alpha+\beta)=\frac{\mathrm{sl}(\alpha)\sqrt{1-\mathrm{sl}^4(\beta)}+\mathrm{sl}(\beta)\sqrt{1-\mathrm{sl}^4(\alpha)}}{1+\mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)}\,.$$

It was derived from the integral relation (12) and therefore is applicable in some vicinity of zero. The formulae (19) and (20) are applicable for any t and are similar to those for the functions $\sin t$ and $\cos t$.

PROPOSITION 5.

$$\mathrm{sl}\,(\alpha+\beta)=\frac{\mathrm{sl}(\alpha)\,\mathrm{sl}'(\beta)+\mathrm{sl}'(\alpha)\,\mathrm{sl}(\beta)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)}\,.$$

PROPOSITION 6.

$$\mathrm{sl}\,(\alpha+\beta)=-\frac{\mathrm{cl}'(\alpha)\,\mathrm{cl}(\beta)+\mathrm{cl}(\alpha)\,\mathrm{cl}'(\beta)}{1+\mathrm{cl}^2(\alpha)\,\mathrm{cl}^2(\beta)}\,.$$

PROPOSITION 7.

$$\mathrm{cl}\,(\alpha+\beta)=\frac{\mathrm{sl}'(\alpha)\,\mathrm{cl}(\beta)+\mathrm{sl}(\alpha)\,\mathrm{cl}'(\beta)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)}\,.$$

The proofs are given in [7].

6. Formulae

We summarize here the main relations for the lemniscatic functions indicating also their counterparts in the theory of elementary trigonometric functions. Proofs are omitted since the formulae are obtained from the basic summation relations using the same type arguments as those used in the theory of elementary trigonometric functions.

$$\mathrm{sl}(\alpha \pm \beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{cl}(\beta) \pm \mathrm{cl}(\alpha)\,\mathrm{sl}(\beta)}{1 \mp \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\mathrm{sl}(\alpha \pm \beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{sl}'(\beta) \pm \mathrm{sl}'(\alpha)\,\mathrm{sl}(\beta)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\sin'\beta \pm \sin'\alpha\sin\beta$$

$$\mathrm{sl}(\alpha \pm \beta) = \mp\frac{\mathrm{cl}(\alpha)\,\mathrm{cl}'(\beta) \pm \mathrm{cl}'(\alpha)\,\mathrm{cl}(\beta)}{1 + \mathrm{cl}^2(\alpha)\,\mathrm{cl}^2(\beta)}$$

$$\sin(\alpha \pm \beta) = \mp(\cos\alpha\cos'\beta \pm \cos'\alpha\cos\beta)$$

$$\mathrm{cl}(\alpha \pm \beta) = \frac{\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta) \mp \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)}{1 \pm \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\mathrm{cl}(\alpha \pm \beta) = \frac{\mathrm{sl}'(\alpha)\,\mathrm{cl}(\beta) \pm \mathrm{sl}(\alpha)\,\mathrm{cl}'(\beta)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)}$$

$$\cos(\alpha \pm \beta) = \sin'\alpha\cos\beta \pm \sin\alpha\cos'\beta$$

$$\mathrm{sl}(\alpha) \pm \mathrm{sl}(\beta) = \frac{2\,\mathrm{sl}\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{sl}'\left(\frac{\alpha\mp\beta}{2}\right)}{1 + \mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\sin'\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\mathrm{sl}(\alpha) \pm \mathrm{sl}(\beta) = \frac{-2\,\mathrm{cl}'\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha\mp\beta}{2}\right)}{1 + \mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\sin\alpha \pm \sin\beta = -2\cos'\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\mathrm{cl}(\alpha)+\mathrm{cl}(\beta)=\frac{2\,\mathrm{cl}\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}'\left(\frac{\alpha-\beta}{2}\right)}{1+\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin'\left(\frac{\alpha-\beta}{2}\right)$$

$$\mathrm{cl}(\alpha)-\mathrm{cl}(\beta)=\frac{2\,\mathrm{cl}'\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}\left(\frac{\alpha-\beta}{2}\right)}{1+\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\cos\alpha-\cos\beta=2\cos'\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\begin{aligned}\mathrm{sl}(\alpha)\pm\mathrm{sl}(\beta)&=2\,\mathrm{sl}\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha\mp\beta}{2}\right)\frac{1+\mathrm{sl}^2\left(\frac{\alpha\mp\beta}{2}\right)}{1+\mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}\\&=2\,\mathrm{sl}\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha\mp\beta}{2}\right)\frac{1+\mathrm{cl}^2\left(\frac{\alpha\pm\beta}{2}\right)}{1+\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)}\end{aligned}$$

$$\sin\alpha\pm\sin\beta=2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\begin{aligned}\mathrm{cl}(\alpha)+\mathrm{cl}(\beta)&=2\,\mathrm{cl}\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)}{1+\mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)}\\&=2\,\mathrm{cl}\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}{1+\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)}\end{aligned}$$

$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\begin{aligned}\mathrm{cl}(\alpha)-\mathrm{cl}(\beta)&=-2\,\mathrm{sl}\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)}{1+\mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}\\&=-2\,\mathrm{sl}\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)}{1+\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)}\end{aligned}$$

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)}{1 - \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)}{\mathrm{sl}^2(\alpha) + \mathrm{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{sl}'(\alpha)}{1 + \mathrm{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\sin'\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)\big(1 + \mathrm{sl}^2(\alpha)\big)}{1 + \mathrm{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{-2\,\mathrm{cl}(\alpha)\,\mathrm{cl}'(\alpha)}{1 + \mathrm{cl}^4(\alpha)}$$

$$\sin 2\alpha = -2\cos\alpha\cos'\alpha$$

$$\mathrm{sl}(2\alpha) = \frac{2\,\mathrm{sl}(\alpha)\,\mathrm{cl}(\alpha)\big(1 + \mathrm{cl}^2(\alpha)\big)}{1 + \mathrm{cl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\mathrm{cl}(2\alpha) = \frac{\mathrm{cl}^2(\alpha) - \mathrm{sl}^2(\alpha)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\alpha)}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\mathrm{cl}(2\alpha) = \frac{\mathrm{sl}'(\alpha)\,\mathrm{cl}(\alpha) + \mathrm{sl}(\alpha)\,\mathrm{cl}'(\alpha)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\alpha)}$$

$$\cos 2\alpha = \sin'\alpha\cos\alpha + \sin\alpha\cos'\alpha$$

$$\begin{aligned}\mathrm{sl}(\alpha)\,\mathrm{sl}(\beta) &= \frac{\mathrm{cl}(\alpha-\beta) - \mathrm{cl}(\alpha+\beta)}{2}\,\frac{1 + \mathrm{cl}^2(\alpha)\,\mathrm{sl}^2(\beta)}{1 + \mathrm{cl}^2(\alpha)}\\ &= \frac{\mathrm{cl}(\alpha-\beta) - \mathrm{cl}(\alpha+\beta)}{2}\,\frac{1 + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)}{1 + \mathrm{cl}^2(\beta)}\end{aligned}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\sin 2\alpha = 2\sin\alpha\sin'\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\sin 2\alpha = -2\cos\alpha\cos'\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \sin'\alpha\cos\alpha + \sin\alpha\cos'\alpha$$

$$= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\begin{aligned}\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta) &= \frac{\mathrm{cl}(\alpha+\beta)+\mathrm{cl}(\alpha-\beta)}{2}\,\frac{1+\mathrm{cl}^2(\alpha)\,\mathrm{sl}^2(\beta)}{1+\mathrm{sl}^2(\beta)}\\ &= \frac{\mathrm{cl}(\alpha+\beta)+\mathrm{cl}(\alpha-\beta)}{2}\,\frac{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)}{1+\mathrm{sl}^2(\alpha)}\end{aligned}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

$$\begin{aligned}\mathrm{sl}(\alpha)\,\mathrm{cl}(\beta) &= \frac{\mathrm{sl}(\alpha+\beta)+\mathrm{sl}(\alpha-\beta)}{2}\,\frac{1+\mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)}{1+\mathrm{sl}^2(\beta)}\\ &= \frac{\mathrm{sl}(\alpha+\beta)+\mathrm{sl}(\alpha-\beta)}{2}\,\frac{1+\mathrm{cl}^2(\alpha)\,\mathrm{cl}^2(\beta)}{1+\mathrm{cl}^2(\alpha)}\end{aligned}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

$$\frac{1+\mathrm{sl}^2(\beta)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)} = \frac{1+\mathrm{cl}^2(\alpha)}{1+\mathrm{cl}^2(\alpha)\,\mathrm{cl}^2(\beta)} \qquad \frac{1+\mathrm{sl}^2(\alpha)}{1+\mathrm{sl}^4(\alpha)} = \frac{1+\mathrm{cl}^2(\alpha)}{1+\mathrm{cl}^4(\alpha)}$$

$$\frac{1+\mathrm{sl}^2(\beta)}{1+\mathrm{cl}^2(\alpha)\,\mathrm{sl}^2(\beta)} = \frac{1+\mathrm{sl}^2(\alpha)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)} \qquad \frac{1+\mathrm{cl}^2(\beta)}{1+\mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)} = \frac{1+\mathrm{cl}^2(\alpha)}{1+\mathrm{cl}^2(\alpha)\,\mathrm{sl}^2(\beta)}$$

REMARK 3. *The last four formulae seem to have no analogues in the elementary trigonometry. They can be proved easily by using the relations*

$$\mathrm{sl}^2(\alpha) = \frac{1-\mathrm{cl}^2(\alpha)}{1+\mathrm{cl}^2(\alpha)}, \qquad \mathrm{cl}^2(\alpha) = \frac{1-\mathrm{sl}^2(\alpha)}{1+\mathrm{sl}^2(\alpha)},$$

which follow from the identity $\mathrm{sl}^2(\alpha) + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(t) + \mathrm{cl}^2(t) = 1$.

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Piezīmes par lemniskātiskajām funkcijām

Kopsavilkums

Uzrādīta virkne formulu, kas saista lemniskātiskās funkcijas $\operatorname{sl} t$ *un* $\operatorname{cl} t$, *kuras apmierina diferenciālvienādojumu sistēmu* $\frac{x'}{1+x^2} = y$, $\frac{y'}{1+y^2} = -x$, *kā arī Emdena–Faulera vienādojumu* $x'' = -2x^3$. *Apskatīta lemniskātisko funkciju teorijas līdzība ar elementāro trigonometrisko funkciju teoriju, uzrādot formulas lemniskātiskām funkcijām un to analogus funkcijām* $\sin t$ *un* $\cos t$. *Atrasti vairāki ekvivalenti saskaitīšanas teorēmas formulējumi funkcijai* $\operatorname{sl} t$, *no kuriem daži šķiet līdz šim nav tikuši apskatīti. Izklāstā tiek izmantota Jakobi eliptisko funkciju teorija.*