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# ON A METHOD FOR PROOFS OF SOME INEQUALITIES BY MEANS OF DEFINITE INTEGRALS

RAITIS OZOLS

*University of Latvia*

Zeļļu iela 8, Rīga, LV – 1002, Latvija

E-mail: `sm01022@lanet.lv`

In this work we propose how to construct and to prove several inequalities between functions. For functions $f$ and $g$ which are defined at an interval (finite or infinite) $A \subset \mathbb{R}$, we introduce the following notations:

$$f(x) >=> g(x) \text{ at } x = x_0 \text{ when } x \in A \Longleftrightarrow \text{ for all } x \in A \quad \begin{cases} f(x) > g(x), & \text{if } x > x_0 \\ f(x) = g(x), & \text{if } x = x_0 \\ f(x) > g(x), & \text{if } x < x_0 \end{cases}$$

$$f(x) <=> g(x) \text{ at } x = x_0 \text{ when } x \in A \Longleftrightarrow \text{ for all } x \in A \quad \begin{cases} f(x) > g(x), & \text{if } x > x_0 \\ f(x) = g(x), & \text{if } x = x_0 \\ f(x) < g(x), & \text{if } x < x_0 \end{cases}$$

For example, it is true that $x - 1 >=> \ln x$ at $x = 1$ when $x > 0$, $\tan x <=> x$ at $x = 0$ when $x \in \ -\frac{\pi}{2}; \frac{\pi}{2}\ $, etc.

We use the following statements for any integrable functions $f$ and $g$:

$$\text{if } f(x) >=> g(x) \text{ for } x = x_0 \text{ then } \int_{x_0}^{x} f(t)\,dt <=> \int_{x_0}^{x} g(t)\,dt \text{ for } x = x_0, \tag{1}$$

$$\text{if } f(x) <=> g(x) \text{ for } x = x_0 \text{ then } \int_{x_0}^{x} f(t)\,dt >=> \int_{x_0}^{x} g(t)\,dt \text{ for } x = x_0. \tag{2}$$

By applying statements (1) and (2), algebraic transformations and substitutions we can prove interesting inequalities. Here are some examples:

1. $\sin x <=> x\sqrt[3]{\cos x}$ at $x = 0$ when $x \in \ -\frac{\pi}{2}; \frac{\pi}{2}\ $;
2. $\sqrt{2}\arctan\frac{x}{\sqrt{2}} <=> \int_0^x e^{-\frac{t^2}{2}}\,dt <=> \sin x$ at $x = 0$ when $x \in \mathbb{R}$;
3. $\underbrace{\sin\sin\cdots\sin}_{n \text{ times}} x <=> \frac{\sin\sqrt{n}x}{\sqrt{n}}$ at $x = 0$ when $x \in \left[-\frac{\pi}{\sqrt{n}}; \frac{\pi}{\sqrt{n}}\right]$, $n \geq 2$;
4. $x^{n^2} >=> T_n(x)$ at $x = 1$ when $x \geq \cos\frac{\pi}{n}$ where $T_n(x)$ is the $n$ th Chebyshev's polynomial of the first kind, $n \geq 2$;
5. $\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+x}{\sqrt{2}-x} + \frac{x}{2-x^2} <=> \tan x$ at $x = 0$ when $x \in (-\sqrt{2}; \sqrt{2})$.

These and other obtained inequalities can be used in several areas of mathematics, for example, in probability theory and automata theory.