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ACTION OF ALGEBRA AUTOMORPHISMS ON ITS MODULES AND FAMILIES OF INDECOMPOSABLE MODULES

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The classic representation/module theory of associative algebras deals with the module categories over algebras with their algebra structures having been fixed once and for all. Modules isomorphism classes are orbits of the set of objects of the finitely generated module category $mod(A)$ under the conjugation action. In this paper we propose a change of this paradigm by introducing an additional action on the category of modules - the action of the automorphism group of the algebra. The orbits of this action define an equivalence relation which is coarser than the standard isomorphism relation. This equivalence relation allows us to study representations of the algebra up to algebra automorphisms which may give nontrivial new results for algebras of wild representation type. Some parameters of multiparameter families of indecomposable modules can be attributed to this action. We show that one parameter indecomposable module families for several tame algebras are closely related to newly defined module equivalence classes. This observation indicates a possibility of a characterization of the finite/tame/wild trichotomy in terms of the structure of the algebra automorphism groups.

Let k be a field, A an associative finite dimensional k-algebra. All modules considered in this paper are left modules. If $A =< X|R >$ where X is the set of generators and R is the set of relations then the image of X under the algebra homomorphism defining the A-module M we call the set of A-action generators for M. By an A-automorphism we mean an endomap of A which preserves the algebra structure of A.

Definition 1. We call two A-modules M_1 and M_2 twisted isomorphic (T-isomorphic, denoted by $M_1 \approx M_2$) provided there exists an A-automorphism $f : A \to A$ and a k-linear isomorphism (twisted isomorphism or T-isomorphism) $\varphi : M_1 \to M_2$ such that for every $a \in A$ we have $f(a) \circ \varphi = \varphi \circ a$. Suppose we are given $f \in Aut(A)$ and a A-module M with the list of A-action generators $(x_1, \dots, x_n)$. We define $f(M)$ (M twisted by f) as the module with the same underlying space and the action generators $(f(x_1), \dots, f(x_n))$. We call the set $\bigcup_{f \in Aut(A)} f(M)$ the T-orbit of M.

Proposition 2. *T-isomorphism is an equivalence relation on $Obj(mod(A))$ which is coarser than the standard module isomorphism equivalence relation. The map $Obj(mod(A)) \to Obj(mod(A))$ defined by $M \to f(M)$ is a group action which preserves module indecomposability.*

We now start to assume that k is algebraically closed. By a family of indecomposable modules we mean a subset of isomorphism classes of $Obj(mod(A))$ continuously depending on arguments taking values in an open subset of k^r for some $r \in \mathbf{N}$, r is said to be the number of parameters of the family. We show that for several classic tame algebras one-parameter families of indecomposable finitely generated modules are T-orbits or their subsets.

First we consider the case $A_1 = k[X]$. Modules in one-parameter families of indecomposable finitely generated A_1-modules are isomorphic to modules with X action matrices being Jordan blocks $J(\lambda, n)$ where $n \in \mathbf{N}$ and $\lambda \in k$. The next case is $A_2 = k[X,Y]/(X,Y)^2$. Modules in one-parameter families of indecomposable finitely generated A_2-modules are isomorphic to modules in the families $\{K(\lambda, n)\}_{\lambda \in k \bigcup \lambda=\infty}$ where modules $K(\lambda, n)$ are given by action generators $K_X(\lambda, n)$ and $K_Y(\lambda, n)$ for X and Y, respectively, which can be chosen as $K_X(\lambda, n) = \begin{pmatrix} O_n & O_n \\ E_n & O_n \end{pmatrix}$, $K_Y(\lambda, n) = \begin{pmatrix} O_n & O_n \\ J(\lambda, n) & O_n \end{pmatrix}$ for any $\lambda \in k$ and additionally $K_X(\infty, n) = \begin{pmatrix} O_n & O_n \\ J(0, n) & O_n \end{pmatrix}$, $K_Y(\infty, n) = \begin{pmatrix} O_n & O_n \\ E_n & O_n \end{pmatrix}$. For more details about finitely generated A_2-modules see [1]. The last case of the tame type is the algebra $A_3 = k[X,Y]/(X^2, Y^2, (XY)^k X^{\epsilon_1}, (YX)^k Y^{\epsilon_2})$ which is related to group algebras for dihedral 2-groups. Modules in one-parameter families of indecomposable A_3-modules are isomorphic to modules in the families $\{B(w, \lambda, m)\}_{\lambda \in k, \lambda \neq 0}$ where modules $B(w, \lambda, m)$ are given by action generators $B_X(w, \lambda, m)$ and $B_Y(w, \lambda, m)$ (of size nm) for X and Y, respectively, which can be chosen as

$$B_X(w, \lambda, m) = \begin{pmatrix} B_X(w,\lambda) & O_n & O_n & \dots & O_n \\ O_n & B_X(w,\lambda) & O_n & \dots & O_n \\ O_n & O_n & B_X(w,\lambda) & \dots & O_n \\ \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & B_X(w,\lambda) \end{pmatrix}$$

and

$$B_Y(w, \lambda, m) = \begin{pmatrix} B_Y(w,\lambda) & O_n & O_n & \dots & O_n \\ O_n + e_{nn} & B_Y(w,\lambda) & O_n & \dots & O_n \\ O_n & O_n + e_{nn} & B_Y(w,\lambda) & \dots & O_n \\ \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & B_Y(w,\lambda) \end{pmatrix}$$

and the matrices $B_X(w, \lambda)$, $B_Y(w, \lambda)$ of size n are determined by the oriented edge labelled graph isomorphism class of an admissible oriented edge labelled cycle w and parameter $\lambda \in k$ along the lines of [2].

Proposition 3. *For any $n \in \boldsymbol{N}$ the one-parameter family $\{J(\lambda, n)\}_{\lambda \in k}$ of A_1-modules is a T-orbit. For any $n \in \boldsymbol{N}$ and any $\lambda \in k$ the one parameter family $\{K(\lambda, n\}_{\lambda \in K \bigcup \lambda=\infty}$ of A_2-modules is a T-orbit. For any $m \in \boldsymbol{N}$ and any admissible word w the one-parameter family $\{B(w, \lambda, m\}_{\lambda \in k, \lambda \neq 0}$ is a subset of a T-orbit.*

Finally we give an example of a two-parameter modules family over a wild algebra which is a subset of a T-orbit. Let $A = k[X,Y,Z]/(X,Y,Z)^2$. A is known to be a minimal wild algebra, i.e. all its subalgebras are of tame or finite representation type (compare with [3]). Consider a family $\{C(\alpha, \beta)\}_{\alpha \in k, \beta \in k, \alpha \neq 0, \beta \neq 0}$ of nonisomorphic indecomposable A-modules of dimension 2 given by action generators $C_X(\alpha, \beta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C_Y(\alpha, \beta) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$, $C_Z(\alpha, \beta) = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$. One can see that $C(\alpha, \beta) \approx C(1, 1)$ and thus the family $\{C(\alpha, \beta)\}_{\alpha \in k, \beta \in k, \alpha \neq 0, \beta \neq 0}$ is a subset of a T-orbit.

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