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ON ZENODOURUS' PROBLEM

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Short historical information. Some remarkable theorems on maximum areas are attributed to Zenodorus (about 300 AD-200 AD). Zenodorus treatise *On isometric figures* has been lost. Fortunately fourteen propositions had been preserved by Pappus (284 AD-305 AD) and Theon (about 335-405) of Alexandria. Here are the most noteworthy of them: Of isoperimetrical, regular polygons, the circle has a greater area than any regular polygon of n sides, the regular is the greatest; of all solids having surfaces equal in area, the sphere has the greatest volume. The essential contribution towards a rigorous proof of optimality of the circle was given in 1841 and is due to Jacob Steiner (1796-1863). Oskar Perron of Tubingen pointed out in 1913 the fallacy of this proof. The first rigorous proof by means of geometry was given by F. Edler (in 1882) [2]. In accordance with [4] *The first rigorous proofs, which used concepts in analysis and calculus of variations were given by F. Edler and by Weierstrass. As of mid 1960's, the question of finding a simple geometric proof was, according to literature, widely believed to be open. However, there have been some not so simple proofs of Eastern European origin that lie within elementary geometry. See, for example* [3].

The proof of optimality or regular n-gone using trigonometric sums but not limits is given in [1]. Attention is focused on elementary proofs of Zenodorus' problem (ZP) of optimality of regular polygons and especially for on a pentagon in this report. However, one can find some new nuances in solving ZP although it is classics of mathematics polished by many outstanding mathematicians. Using Cauchy's inequality one can easy prove a ZP for a hexagon following the scheme:

$$A(a_1, a_2, a_3, a_4, a_5, a_6) \leq A(a, a, b, b, c, c) \leq 2A(a, b, c) \leq A(R),$$

where $A(R)$ - area of the regular isoperimetric hexagon R, or using (provable by an elementary means) general inequality

$$A(a_1, a_2, \dots, a_n) \leq A(a, a, \dots, a)$$

which is valid for an arbitrarily and some isoperimetric equilateral n-gon. However, I am not aware of the existence of such a simple proof for a pentagon. The proposed solution of ZP for a pentagon is based on the determination by elementary methods the maximum value of the following function [see: http://www.liis.lu.lv]

$$A = \frac{x}{2}\sqrt{4a^2 - x^2} + \frac{a}{4}\sqrt{4x^2 - a^2}.$$

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